Estimation of Parameters for RICHARDS Model

TATSURO AKAMINE

Abstract

AKAMINE (1986)'s BASIC program by MARQUARDT's method was rewritten for RICHARDS model and its expanded model by the periodic function. For 0.9—1.1 the "LOG" function is corrected by TAYLOR series. Data estimated to be negative are cut off. AIC judges the effect of adding \( n \) to the parameters. RICHARDS model is not so important in practice but it is important theoretically.

Key words  RICHARDS, MARQUARDT, TAYLOR series, AIC, BASIC program

I. Introduction

AKAMINE (1986) estimated parameters by MARQUARDT's method for von BERTalanffy, logistic and Gompertz models and their expanded models by the periodic function. RICHARDS model includes these three models. In this paper, estimation of parameters for RICHARDS model and correction of the "LOG" function in the calculation will be described.

II. RICHARDS model

1. Model

RICHARDS model is defined in the differential equation as follows:

\[
\frac{dl}{dt} = K t \left( \frac{l}{l_{\infty}} \right)^n
\]  \quad (2.1)

Then let

\[
v = \left( \frac{l_{\infty}}{l} \right)^n - 1
\]  \quad (2.2)

Therefore,

\[
\frac{dv}{dt} = -\frac{l_{\infty}^n}{l^{n+1}} \frac{dl}{dt}
\]  \quad (2.3)

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Substitution of (2.1) and (2.2) into (2.3) gives

\[ \frac{dv}{dt} = -Kv. \]  

(2.4)

A general solution of this differential equation is

\[ v = e^h, \quad h = -Kt + c, \quad c : \text{integral constant.} \]  

(2.5)

Substitution of (2.5) into (2.2) gives

\[ l = \frac{l_\infty}{1 + ne^h}, \quad h = -Kt + c. \]  

(2.6)

This is the general solution of Richards model.

The initial condition :

\[ \text{when } t=t_0, \quad l = \frac{l_\infty}{1 + n}, \]  

(2.7)

gives the particular solution :

\[ l = \frac{l_\infty}{1 + n}, \quad h = -K(t-t_0). \]  

(2.8)

On the other hand, the initial condition :

\[ \text{when } t=0, \quad l = l_0 \]  

(2.9)

gives the particular solution :

\[ l = \frac{l_\infty}{1 + (p^n-1)e^h}, \quad h = \frac{l_\infty}{l_0}, \quad h = -Kt. \]  

(2.10)

Note that

\[ \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = \lim_{n \to 0} (1 + nx)^{1/n} = e^x, \]  

(2.11)

\[ \lim_{n \to 0} \frac{y^n - 1}{n} = \ln y. \]  

(2.12)

Then (2.8) corresponds to three models as follows :

\[ \begin{cases} 
  n = -1 : \text{Von Bertalanffy model} \\
  n \to 0 : \text{Gompertz model} \\
  n = 1 : \text{logistic model} 
\end{cases} \]

Relation of \( l_0 \) and \( l_0 \) is

\[ l_0 = \frac{1}{K} \ln \frac{b^n - 1}{n}, \quad \text{when } n \to 0, \quad l_0 = \frac{1}{K} \ln (\ln b). \]  

(2.13)

Although (2.9) is more general than (2.7) as an initial condition, (2.7) is usually used in fishery population dynamics and is easier to treat in calculation. (2.13) combines (2.8) and (2.10).
2. Property

Setting \( l'' = 0 \), we find that

\[
1 - (n+1) \left( \frac{l}{l_\infty} \right)^n = 0.
\]  
(2.14)

When \( n > -1 \), this equation has a solution corresponding to (2.7). Namely, when \( n > -1 \), \( t_0 \) is an inflection point.

Outlines of this model are as follows: First, (2.10) gives

\[
\lim_{n \to \infty} l = l_0,
\]  
(2.15)

\[
\lim_{n \to -\infty} l = \begin{cases} 
  l_0 & (t = 0) \\
  l_\infty & (t > 0) 
\end{cases}
\]  
(2.16)

These are shown in Fig. 1. On the other hand, (2.8) gives

\[
\lim_{n \to \infty} l = l_\infty,
\]  
(2.17)

\[
\lim_{n \to -\infty} l = \begin{cases} 
  0 & (t = t_0') \\
  l_\infty & (t > t_0') 
\end{cases}, \quad t_0' = t_0 + \frac{\ln(-n)}{K}.
\]  
(2.18)

Namely, (2.8) has intersection \( t_0' \) with the transverse axis \( (l = 0) \) when \( n < 0 \). These are shown in Fig. 2. When \( n < 0 \), (2.8) is rewritten as

\[
l = l_\infty (1 - e^h)^m, \quad h = -K(t - t_0'), \quad m = -\frac{1}{n}.
\]  
(2.8')

**Richards model**

![Richards model graph](image)

**Fig. 1. Richards model**: \( l = \frac{l_\infty}{l_0}, \quad p = \frac{l_\infty}{l_0}, \quad h = -Kt. \)

\( l_\infty = 100, \quad K = 0.2, \quad l_0 = 10, \quad n = -\infty, -5, -2, -1, 0, 1, 2, 5, \infty. \)
This is called the generalized von Bertalanffy model. Specifically, it is generally used as a body weight growth model when $n = -1/3$.

3. Expanded model by the periodic function

This expansion is defined as follows:

$$K \rightarrow Kf(t), \quad f(t+1) = f(t)$$

(Akamine (1986) presented two models defined as follows in (2.4) :

$$\frac{dv}{dt} = -Kf(t)v \quad \text{(type-1)}$$

$$\frac{dv}{dt} = -Kf(t)eh \quad \text{(type-2)}$$

In this paper, only the type-1 model is discussed because the type-1 is more natural than the type-2. The particular solution of (2.20) with the initial condition as (2.7) is

$$l = \frac{l_{\infty}}{(1 + neh)^{1/n}}, \quad h = -K(F(t) - F(t_0)), \quad F'(t) = f(t).$$

(Akamine (1986) used the following periodic function.

$$f(t) = \frac{1 + a}{2} + \frac{1 - a}{2} \cos 2\pi(t - t_0)$$
Parameters for Richards Model

\[ F(t) = \frac{1+a}{2} t + \frac{1-a}{4\pi} \sin 2\pi(t-t_i) \]  
(2.24)

The inflection points are given by the following equation.

\[ h'^2(1-eh) + h''(1+neh) = 0 \]  
(2.25)

This equation is the general equation of equation (33), (53) and (63) of Akamine (1986) and can be solved by Newton’s method.

III. Estimation of parameters

1. Marquardt’s method

This is the expanded Newton’s method. Let \( Y \) be the objective function, \( \theta \) be parameters. It is expressed as follows in the case of searching the minimal point.

\[(H + \lambda I) \mathbf{J}\theta = g, \quad H = \left( \frac{\partial^2 Y}{\partial \theta_i \partial \theta_j} \right), \quad g = -\frac{\partial Y}{\partial \theta} \]  
(3.1)

\( H \): Hessian matrix
\( I \): unit matrix
\( g \): gradient vector

\( \lambda \) is the control factor of convergence. When \( \lambda \to \infty \) it approaches the steepest descent method: \( \mathbf{J}\theta = (1/\lambda)g \). On the other hand, when \( \lambda \to 0 \) it approaches Newton’s method: \( H \mathbf{J}\theta = g \). Therefore, let \( \lambda \) be large at first, and make it small step by step to get the solution. In this paper, the simplest method is used. When \( JY < 0 \) let \( \lambda_{\text{new}} = \lambda_{\text{old}}/2 \) to continue the calculation, when \( JY \geq 0 \) let \( \lambda_{\text{new}} = \lambda_{\text{old}} \times 2 \) and try again the same iterative routine. When \( JY \geq 0 \) after 10 times enlarging \( \lambda \) continuously, we determine it to be the solution to end the calculation.

Scaling of the parameters is necessary because Marquardt’s method is like to the steepest descent method at first. Scaling is defined by a diagonal matrix \( S \). Then (3.1) becomes

\[(S^{-1}HS^{-1} + \lambda I) S \mathbf{J}\theta = S^{-1}g.\]

Cleaning this equation, we have

\[(H + \lambda S^2) S \mathbf{J}\theta = g.\]  
(3.2)

Generally, we use

\[S^2 = \text{diag}H.\]  
(3.3)

\( \text{diag}A \): diagonal matrix composed of only the diagonal elements of \( A \)

Then it is sufficient to enlarge the diagonal elements of \( H \) by a factor \((1+\lambda)\). If \( H \) has a negative part of its diagonal elements, let \( \lambda > 1 \) and enlarge that part by a factor \((\lambda - 1)\) and the other part by a factor \((1+\lambda)\).

The objective function is the weighted least-squares method.
\[ Y = \sum_{k=1}^{N} \left( \frac{l - l_{k}^{\circ}}{a_{k}^{2}} \right)^{2}, \]  
(3.4)

\( N \) : number of data

\( l_{k}^{\circ}, a_{k}^{\circ} \) : data

Then it leads to

\[ \frac{\partial Y}{\partial l} = \sum_{k} \frac{2(l - l_{k}^{\circ})}{a_{k}^{2}}, \quad \frac{\partial^{2} Y}{\partial l^{2}} = \sum_{k} \frac{2}{a_{k}^{2}}, \]  
(3.5)

\[ \frac{\partial^{2} Y}{\partial \theta_{i} \partial \theta_{j}} = \frac{\partial^{2} Y}{\partial l^{2}} \frac{\partial l}{\partial \theta_{i}} \frac{\partial l}{\partial \theta_{j}} + \frac{\partial Y}{\partial l} \frac{\partial l}{\partial \theta_{i}} \frac{\partial l}{\partial \theta_{j}}. \]  
(3.6)

Because the second term of (3.6) contributes a little, it is omitted generally. Then we get

\[ \frac{\partial^{2} Y}{\partial \theta_{i} \partial \theta_{j}} = \frac{\partial^{2} Y}{\partial l^{2}} \frac{\partial l}{\partial \theta_{i}} \frac{\partial l}{\partial \theta_{j}} + \frac{\partial Y}{\partial l} \frac{\partial l}{\partial \theta_{i}} \frac{\partial l}{\partial \theta_{j}} = \sum_{k=1}^{N} \frac{2}{a_{k}^{2}} \frac{\partial l}{\partial \theta_{i}} \frac{\partial l}{\partial \theta_{j}}. \]  
(3.7)

Therefore, the diagonal elements of \( H \) are always positive.

2. Partial differentiation by parameters

Parameters of Richards model (2.8) and (2.22) are \( l_{\infty}, K, l_{0}, t_{1}, a \) and \( n \). Concrete expressions of \( \partial l/\partial \theta \) are as follows:

\[ \frac{\partial l}{\partial l_{\infty}} = \frac{l}{l_{\infty}}, \]  
(3.8)

\[ \frac{\partial l}{\partial \theta} = -l \frac{\theta}{1 + n \theta} \frac{\partial h}{\partial \theta}, \quad \theta = K, l_{0}, t_{1}, a. \]  
(3.9)

Where for (2.8)

\[ \frac{\partial h}{\partial K} = -(t - t_{0}) \]

\[ \frac{\partial h}{\partial t_{0}} = K. \]

Where for (2.22)

\[ \frac{\partial h}{\partial K} = -\left\{ F(t) - F(t_{0}) \right\} \]

\[ \frac{\partial h}{\partial t_{0}} = Kf(t_{0}) \]

\[ \frac{\partial h}{\partial t_{1}} = -K \left\{ \frac{\partial F(t)}{\partial t_{1}} - \frac{\partial F(t_{0})}{\partial t_{1}} \right\} \]

\[ \frac{\partial F(t)}{\partial t_{1}} = -\frac{1-a}{2} \cos 2\pi(t - t_{1}) \]

\[ \frac{\partial h}{\partial a} = -K \left\{ \frac{\partial F(t)}{\partial a} - \frac{\partial F(t_{0})}{\partial a} \right\} \]

\[ \frac{\partial F(t)}{\partial a} = \frac{1}{2} t - \frac{1}{4\pi} \sin 2\pi(t - t_{1}). \]
Parameters for Richards Model

For $n$

$$\frac{\partial l}{\partial n} = l \left( 1 + nx \right) \left( \frac{1}{n} \ln(1+nx) - \frac{x}{1+nx} \right) > 0, \quad x = e^h. \quad (3.10)$$

$$\lim_{n \to 0} \frac{\partial l}{\partial n} = \frac{x^2}{2} > 0.$$

The sign of (3.10) is apparent in Fig. 2.

3. Correction of the "LOG" function

When $n \to 0$ it is difficult to calculate (2.8), (2.22) and (3.10) precisely. When we use a high precision computer, it is sufficient to be careful only for $n=0$. The probability of $n$ being 0 is so low for normal data that we can ignore this case. But, when we use a low precision computer, this problem is important because the precision of its "LOG" function is too low.

Takahashi (1974) and Hitotsumatsu (1981) suggested that computers treat the calculation of power as follows:

$$x^y = \text{EXP}(y \ast \text{LOG}(x)).$$

Although the "EXP" function is high precision, the "LOG" function is low precision. The values of $\ln(1+n)$ for a personal computer PC9801F (NEC, N98-BASIC) and a hand-held calculator FX502P (CASIO) are shown in Table 1. The FX502P result is correct and the PC9801F result is not correct because of Taylor series as follows:

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \ldots. \quad (3.11)$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>PC9801F</th>
<th>FX502P</th>
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</thead>
<tbody>
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<td>11.5129</td>
<td>11.51293546</td>
</tr>
<tr>
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<td>9.21044</td>
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<td>4.61512516</td>
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<td>2.397685272</td>
</tr>
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<td>0.693147</td>
</tr>
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<td>0.1</td>
<td>9.53102 × 10^{-2}</td>
<td>9.5310179 × 10^{-2}</td>
</tr>
<tr>
<td>0.01</td>
<td>9.95025 × 10^{-3}</td>
<td>9.95030853 × 10^{-3}</td>
</tr>
<tr>
<td>0.001</td>
<td>9.99446 × 10^{-4}</td>
<td>9.99500333 × 10^{-4}</td>
</tr>
<tr>
<td>10^{-4}</td>
<td>9.99405 × 10^{-5}</td>
<td>9.99950033 × 10^{-5}</td>
</tr>
<tr>
<td>10^{-5}</td>
<td>9.91555 × 10^{-6}</td>
<td>9.99995000 × 10^{-6}</td>
</tr>
<tr>
<td>10^{-6}</td>
<td>9.08925 × 10^{-7}</td>
<td>9.99999500 × 10^{-7}</td>
</tr>
<tr>
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<td>9.99999500 × 10^{-8}</td>
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<td>0</td>
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</tr>
<tr>
<td>10^{-9}</td>
<td>0</td>
<td>9.99999995 × 10^{-10}</td>
</tr>
</tbody>
</table>
And (3.10) has the problem of cancellation. Let

\[ a = \frac{1}{n} \ln(1+nx), \quad b = -\frac{x}{1+nx}. \]

When \( n = 0.1 \), \( x = 0.005 \) it is as follows:

<table>
<thead>
<tr>
<th></th>
<th>PC9801F</th>
<th>FX502P</th>
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<tbody>
<tr>
<td>( a )</td>
<td>4.99826</td>
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<tr>
<td>( b )</td>
<td>4.9975</td>
<td>4.997501249</td>
</tr>
<tr>
<td>( a - b )</td>
<td>0.00076</td>
<td>0.001249167</td>
</tr>
</tbody>
</table>

Thus, the PC9801F gives the wrong values. When \( n \) is small, the number of significant figures drop even in the FX502P values.

In this paper, for (2.8) and (2.22) the following expression is used by (3.11).

\[(1+nx)^{-n} = \exp \left[ x \left\{ \frac{-nx}{2} + \frac{(nx)^2}{3} - \frac{(nx)^3}{4} + \ldots \right\} \right] \quad (3.12)\]

And for (3.10) the following expression is used by (3.11) and

\[ \frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - \ldots \quad (3.13)\]

\[ \frac{\partial f}{\partial n} = \frac{1}{2} x^2 \left\{ 1 - \frac{2}{3} \frac{2nx}{3} - \frac{2}{4} \frac{3(nx)^2}{4} - \frac{2}{5} \frac{4(nx)^3}{5} + \ldots \right\}, \quad x = e^k. \quad (3.14)\]

(3.12) and (3.14) are used when \( |nx| \leq 0.1 \).

**IV. AIC**

When we treat \( n \) as a parameter the number of parameters increases by 1. Therefore, the likelihood increases. But, the confidence area will be enlarged because the correlations of parameters increase. An adequate number of parameters will be presented by AIC (Akaike information criterion):

\[ \text{AIC} = -2 \ln L_{\text{max}} + 2r \quad (4.1) \]

\( L_{\text{max}} \): maximum likelihood
\( r \): number of parameters

For the weighted least-squares method (3.4), \( (l_k - l^*)/\sigma_k^* \) distributes according to \( N(0, 1) \). Then it leads to

\[ L = \prod_{k=1}^{N} P_k = \left( \frac{1}{\sqrt{2\pi}} \right)^N \exp \left( -\frac{1}{2} Y \right), \quad (4.2) \]

\[ \text{AIC} = Y_{\text{min}} + 2r + \text{const.} \quad (4.3) \]

On the other hand, for the least-squares method:
\[ Y^* = \sum_{k=1}^{N} (l-l_k^*)^2 \]  
(4.4)

\((l_l-l_k^*)\) distributes according to \(N(0, \sigma)\). Then it leads to

\[ L = \left(\frac{1}{\sqrt{2\pi} \sigma} \right)^N \exp \left( -\frac{1}{2} \frac{Y^*}{\sigma^2} \right). \]  
(4.5)

The following is used for the \(\sigma^2\) estimate.

\[ \sigma^2 = \frac{Y_{\text{min}}^*}{N-r} \]  
(4.6)

Then it leads to

\[ \text{AIC} = N \ln Y_{\text{min}}^* + 2r + \text{const}. \]  
(4.7)

Where the following approximation is used.

\[ \ln \left(1 - \frac{r}{N} \right) \approx -\frac{r}{N} \quad (r \ll N) \]

The model which minimizes AIC is regarded as the best. Namely, the model which explains the data efficiently with fewer parameters is regarded as best. Generally, AIC may be useful in the condition \(r \ll 2\sqrt{N}\).

V. Computation program

The BASIC programs for FC9801F (NEC) are listed in Appendices B and C. These are the only changing parts from Akamine (1986)'s program 1. When \(n<0\), the data satisfying \(1+nZ\leq0\) are cut off. This is not so important in practice. After line 20000 there is a correction to the "LOG" function. This is not necessary for high precision computers or languages. It is better to check the "LOG" function before use.

VI. Experiments

Akamine (1986)'s data (Table 2) is used for the test of these programs and results are shown in Table 3. Adding \(n\) to the parameters makes \(Y_{\text{min}}\) smaller but AIC larger. Namely, it is not efficient for these data. It is natural because they are made for \(n=-1, 0\) and 1. On the other hand, adding \(l_l\) and \(a\) to the parameters makes AIC smaller. They relate only to oscillation.

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K. Tanaka of Japan Sea Regional Fisheries Research Laboratory for their critical reading of the manuscript.

**Table 2-a.** The data for the experiment (data-1).

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<th>$l_k^c$</th>
<th>$\sigma_k^c$</th>
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**Table 2-b.** The data for the experiment (data-2).

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**Table 2-c.** The data for the experiment (data-3).

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Table 3. Results of experiments. Comparison with Akamine (1986).

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a) number of parameters
b) $\text{AIC} = Y_{\text{min}} + 2r$
c) $k=1$ data is cut off.
References


RICHARDS の式のパラメータ推定

赤塚 達郎

Akamine (1986) の Marquardt 法の BASIC プログラムを Richards の式およびその周期関数による拡張型式を書き換えた。0.90〜1.1において Taylor 級数で“LOG” 関数の修正を行った。推定値が負となるデータは除外した。n をパラメータに加える事の妥当性を AIC で判定した。Richards の式は応用上はあまり重要ではないが、理論上重要である。

Appendix A. Partial differentiation by $n$ for (2.10).

This is as follows:

$$
\frac{\partial l}{\partial n} = \frac{1}{n} \left[ \frac{1}{n} \ln \left\{ 1 + \left( \frac{p^n - 1}{x} \right) x \right\} - \frac{x p^n \ln p}{1 + (p^n - 1)x} \right] - 0, \quad x = e^b.
$$

$$
\lim_{n \to 0} \frac{\partial l}{\partial n} = - (\ln p)^2 \frac{x - x^2}{2} < 0.
$$

The sign is apparent in Fig. 1. These expressions seem to be more difficult to treat than (3.10). The expression of $n \to 0$ is led by (3.11), (3.13) and

$$
\lim_{n \to 0} \frac{y^{n-1}}{n} - \ln y = \frac{(\ln y)^2}{2}.
$$

In addition to

$$
\lim_{n \to 0} \frac{e^x - (1 + nx)^{1/2}}{n} = e^{\frac{x}{2}}.
$$

These are led by the following theory:

When $\lim_{n \to 0} f = 0$ and $\lim_{n \to 0} g = 0$, $\lim_{n \to 0} \frac{f}{g} = \lim_{n \to 0} \frac{f'}{g'}.$
Appendix B. The BASIC program to estimate parameters for Richards model.
Changing parts from program-1 of Akamine (1986).

10 '-----------------------------------------------
20 ' Richards model by Marquardt's method
30 ' by Tatsuro Akamine
40 ' 1987-06-24
50 '-----------------------------------------------
60 '-----------------------------------------------
70 ' Definition of functions
80 '-----------------------------------------------
90 '-----------------------------------------------
100 NP=4
110 DEF FNP=EXP1(-P2(2)*(TIME(K)-P2(3)))
115 DEF FNP2=1+P2(4)*FNP
120 DEF FNDP1=1/PPOWER9
130 DEF FNBL=P2(1)*FNDP1
140 DEF FNDP2=FNBL*FNEP1/FNEP2*(TIME(K)-P2(3))
150 DEF FNDP3=FNBL*FNEP1/FNEP2*P2(2)
160 DEF FNDP4=FNBL*DDL9
170 DEF FNC9801=FNEP2*(1/P2(4))
180 DEF FNC9802=(LOG(FNEP2)/P2(4)-FNEP1/FNEP2)/P2(4)
190 '-----------------------------------------------
200 IF FNEP2<0 THEN PRINT "CANCEL 2 I");;K : GOTO *CSKP2
210 GOSUB *CHECK1
220 GOSUB *CHECK2
230 DIFER(4)=FNDP4
240 NEXT J : NEXT I
250 *CSKP2
260 NEXT K
270 IF FNEP2<0 THEN LPRINT "CANCEL 1 I");;K : GOTO *CSKP1
280 GOSUB *CHECK1
290 *CSKP1
300 IF Richards n ="];P(4)
310 '-----------------------------------------------
320 ' Correction of (1+nx)^1/n and d1/dn
330 '-----------------------------------------------
340 '-----------------------------------------------
350 *CHECK1
360 BRANCH=P2(4)*FNEP1
370 IF ABS(BRANCH)<.1 THEN POWERS=FNP9801 ELSE GOSUB *CORRECT1
380 RETURN
390 *CHECK2
400 BRANCH=P2(4)*FNEP1
410 IF ABS(BRANCH)<.1 THEN DDL9=FNC9802 ELSE GOSUB *CORRECT2
420 RETURN
430 *CORRECT1
440 COR1=-BRANCH
450 COR12=2 : CORD1=1
460 *CORSTART1
470 CORC1=COR1/COR1
480 IF ABS(CORC1)<.000001 THEN *COREND1
490 CORD1=CORD1+CORC
500 COR1=COR1*(-BRANCH) : CORI1=COR1+1
510 GOTO *CORSTART1
520 *COREND1
530 POWERS=EXP1(FNEP1*CORD1)
540 RETURN
550 *CORRECT2
560 COR2=-BRANCH
570 COR22=3 : CORD2=1
580 *CORSTART2
590 CORC2=COR2*(COR2-1)/COR2
600 IF ABS(CORC2)<.0000001 THEN *COREND2
610 CORD2=CORD2+CORC2
620 COR2=COR2*(-BRANCH) : CORI2=COR1+1
630 GOTO *CORSTART2
640 *COREND2
650 DDL9=FNEP1*FNEP1*CORD2/2
660 RETURN
Appendix C. The BASIC program to estimate parameters for Richards model expanded by a periodic function. Changing parts from appendix B.

```
10  '------------------------------------------------------------------------
20  ' Richards model expanded by periodic function
30  ' by Marquardt's method
40  ' by Tatsuro Akamine
50  ' 1987-06-25
60  '------------------------------------------------------------------------

1020 PAI=3.14159265#
1100 NP=6
1110 DEF FNCEP1=EXP(-P2(2)*(FNFT1(TIME(K))-FNFT1(P2(3))))
1115 DEF FNCEP2=1+P2(4)*FNCEP1
1120 DEF FNCEP4=1/POWER9
1130 DEF FNBL =P2(1)*FNCEP1
1140 DEF FNBL2=FNCEP3*(FNFT1(TIME(K))-FNFT1(P2(3)))
1150 DEF FNCEP3=-FNCEP3*P2(2)*FNFT3(P2(3))
1190 DEF FNCEP4=FNBL*DLN9
1200 DEF FNCEP5=FNCEP3*P2(2)+((FNFT4(TIME(K))-FNFT4(P2(3))))
1210 DEF FNCEP6=FNCEP3*P2(2)*((FNFT5(TIME(K))-FNFT5(P2(3))))
1300 DEF FNCEP3=FNBL*FNCEP1/FNCEP2
1400 DEF FNFT1(TM)=FNPS1*TM+FNPS2/2/PAA*SIN(FNMT1(TM))
1410 DEF FNFT3(TM)=FNPS1+FNPS2*COS(FNMT1(TM))
1420 DEF FNFT4(TM)=-FNPS2*COS(FNMT1(TM))
1430 DEF FNFT5(TM)=TM/2-1/4/PAA*SIN(FNMT1(TM))
1450 DEF FNPS1=(1+P2(6))/2
1460 DEF FNPS2=(1-P2(6))/2
1470 DEF FNMT1(TM)=2*PAA*(TM-P2(5))
4102 DIFFER(5)=FNCEP5
4103 DIFFER(6)=FNCEP6
8666 PRINT " T1 =";P(5)
8667 PRINT " A =";P(6)
```