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Expansion of Growth Curves Using a Periodic Function and BASIC Programs by MARQUARDT's Method

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Abstract

The growth curves of VON BERTALANFFY, logistic and GOMPERTZ models were expanded using a periodic function, $f(t+1)=f(t)$. Each model was expanded into $l=l_\infty(1-\exp h_1)$, $l=l_\infty/(1+\exp h_1)$ and $l=l_\infty\exp(-\exp h_1)$ where $h_1=-K\{F(t)-F(t_0)\}$, $F'=f$, $f=(1+a)/2+(1-a)/2\cdot\cos 2\pi(t-t_1)$: $a\leqq f\leqq 1$.

BASIC programs for each model were written by MARQUARDT's method according to AKAMINE (1985). The following subjects were also considered : an expansion into another type, a parameter-error analysis, a comparison with the original model and with WALFORD's graphical method, and a calculation to determine the extreme points of the growth rate. This expansion of the growth curves is useful and the programs are easily applied to other curves.

I. Introduction

For displaying growth curves, VON BERTALANFFY, logistic and GOMPERTZ models are widely used. However, it is often difficult to use such curves for data obtained from short intervals since the growth rates of aquatic organisms are periodically affected by such variables as water temperature. Data is not used effectively because WALFORD's graphical method is mainly used for calculations.

PITCHER and MACDONALD (1973) and PAULY and DAVID (1981) had already expanded growth curves. But these have not been widely used because related methods of calculation were not very useful. On the other hand, CONWAY *et al* (1970) had already used MARQUARDT's method for a logistic model, a FORTRAN program for large computers (only).

This paper has been written to allow its application to any type of growth curve. Two types of expansions using periodic functions are considered. The BASIC programs of MARQUARDT's method were modified according to AKAMINE (1985) and tested using artificial data.

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II. Expansion of VON BERTALANFFY model

1. Modeling of a growth curve

The differential equation of VON BERTALANFFY model is

$$\frac{dl}{dt}=a(l_\infty-l), \qquad \text{where} \qquad a=\text{const.} \qquad \text{——(1)}$$

The integral of equation (1) with an initial condition, when $t=t_0$ let $l=0$, is

$$l=l_\infty(1-\exp h_0), \qquad \text{where} \qquad h_0=-K(t-t_0). \qquad \text{——(2)}$$

This is a "type-0" equation. The differential of equation (2) is

$$\frac{dl}{dt}=a^*\exp h_0, \qquad \text{where} \qquad a^*=Kl_\infty. \qquad \text{——(3)}$$

From equations (1) and (3), if the growth rate changes periodically along with the water temperature, there are two types of models:

$$\frac{dl}{dt}=a(l_\infty-l)f(t) \qquad \text{——(4)}$$

and

$$\frac{dl}{dt}=a^*(\exp h_0)f(t) \qquad \text{——(5)}$$

The period of the water temperature is 1, as follows:

$$f(t+1)=f(t) \qquad \text{——(6)}$$

Let equation (4) be a "type-1" equation and equation (5) be a "type-2" equation. First, consider the type-1 equation. The integral of quation (4) with the same initial condition as the type-0 equation is

$$l=l_\infty(1-\exp h_1), \qquad \text{where} \qquad h_1=-K\{F(t)-F(t_0)\} \qquad \text{——(7)}$$

$$\text{and} \qquad F=\int f dt. \qquad \text{——(8)}$$

The differential of equation (7) is

$$\frac{dl}{dt}=a^*(\exp h_1)f(t). \qquad \text{——(9)}$$

Next, consider the type-2 equation. The integral of equation (5) with the same initial condition as the type-0 equation is

$$l=l_\infty\{1-\frac{G(t)}{G(t_0)}\exp h_0\}=l_\infty(1-\exp h_2), \qquad \text{where} \qquad \text{——(10)}$$

$$h_2=h_0+\ln G(t)-\ln G(t_0)$$

$$\text{and} \qquad G\exp h_0=\int f\exp h_0 dt. \qquad \text{——(11)}$$

A comparison between equation (9) and equation (5) makes it easy to understand the difference between type-1 and type-2 equations.

Now, let's consider G. Equation (11) becomes

$$G'+Gh'_0=fh'_0, \qquad \text{where} \qquad h'_0=-K, \quad f(t+1)=f(t)$$

and

$$G'(t+1)-G'(t)=K\{G(t+1)-G(t)\}.$$

Then,

$$G(t+1)-G(t)=C\exp(Kt) \qquad (C=\text{const.}\geqq 0). \qquad \text{——(12)}$$

$G(n)$ can be written as,

$$G(n)=C\frac{\exp(Kn)-1}{\exp K-1}+G(0).$$

A particular solution of equation (12) is,

$$G(t)=a\exp(Kt)+b,$$

and its general solution of equation (12) is,

$$G(t)=a\exp(Kt)+b+g(t),\quad \text{where}\quad g(t+1)=g(t). \qquad \text{——(13)}$$

The first term on the right side of equation (13) is the constant term $G\exp h_0$, considered to be an integral constant. Therefore, it is natural to let G be

$$G(t+1)=G(t). \qquad \text{——(14)}$$

2. The practical model

Many forms can be used for the periodic function f. In this paper, the simplest function is used:

$$f(t)=\frac{1+a}{2}+\frac{1-a}{2}\cos 2\pi(t-t_1),\quad \text{where}\quad a\leqq f\leqq 1. \qquad \text{——(15)}$$

Therefore,

$$F(t)=\frac{1+a}{2}t+\frac{1-a}{4\pi}\sin 2\pi(t-t_1),\quad \text{and} \qquad \text{——(16)}$$

$$G(t)=\frac{1+a}{2}+\frac{1-a}{2}\,\frac{K}{K^2+4\pi^2}\{K\cos 2\pi(t-t_1)-2\pi\sin 2\pi(t-t_1)\} \qquad \text{——(17)}$$

$$=\frac{1+a}{2}+\frac{1-a}{2}\cos\theta\cos\{\theta+2\pi(t-t_1)\},\quad \text{where}\quad \cos\theta=\frac{K}{\sqrt{K^2+4\pi^2}}.$$

Then, equation (7) becomes essentially the same as the models of PITCHER and MACDONALD(1973) and PAULY and DAVID(1981). Examples were shown in the former study for $a=-1$ and in the latter for $a=0$ (because their calculating methods were not so useful).

3. The calculating method

(1) Outline

In general, NEWTON's method or the steepest descent method is sufficient when the number of parameters is less than or equal to 3. MARQUARDT's method is most appropriate when there are more than 3 parameters. MARQUARDT's method has been adopted in this program, since the number of type-1 and type-2 parameters are both 5. It is also useful for more complicated functions of f.

A weighted least-squares method was adopted for the object function. When the data are (t_1, l_{01}, σ_1),........., (t_n, l_{0n}, σ_n), the object function is

$$Y=\sum_{i=1}^{n}\frac{(l_{0i}-l)^2}{\sigma_i^2}. \qquad \text{——(18)}$$

If $\sigma_i=1$ $(i=1\sim n)$, it becomes a normal least-squares procedure.

This BASIC program has been rewritten according to the program of AKAMINE(1985). but the method for scaling the parameters is according to MARQUARDT(1963).

(2) MARQUARDT's method

MARQUARDT's method is expressed as follows (in the case of searching for the minimal point).

$$(\boldsymbol{H}+\lambda \boldsymbol{I})\boldsymbol{\Delta\theta}=\boldsymbol{g} \qquad (19)$$

$$\boldsymbol{H}=\left(\frac{\partial^2 Y}{\partial\theta_i\partial\theta_j}\right)\doteqdot\left(\frac{\partial^2 Y}{\partial l^2}\frac{\partial l}{\partial\theta_i}\frac{\partial l}{\partial\theta_j}\right)=\frac{\partial^2 Y}{\partial l^2}\,{}^t\!\left(\frac{\partial l}{\partial\boldsymbol{\theta}}\right)\left(\frac{\partial l}{\partial\boldsymbol{\theta}}\right)$$

$$\boldsymbol{g}=-\frac{\partial Y}{\partial\boldsymbol{\theta}}=-\frac{\partial Y}{\partial l}\frac{\partial l}{\partial\boldsymbol{\theta}}$$

$\boldsymbol{I}$: unit matrix
$\boldsymbol{\Delta\theta}$: correction of $\boldsymbol{\theta}$
${}^t\boldsymbol{A}$: transposed matrix of $\boldsymbol{A}$

When λ is large, the method approaches the steepest descent method, as follows :

$$\boldsymbol{\Delta\theta}\doteqdot\frac{1}{\lambda}\boldsymbol{g} \quad . \qquad (20)$$

On the other hand, when λ is small it approaches NEWTON's method, as follows:

$$\boldsymbol{H\Delta\theta}\doteqdot\boldsymbol{g} \quad . \qquad (21)$$

The steepest descent method is stable but has a slow convergence; NEWTON's method has the opposite characteristics. Therefore, in order to obtain a good convergence, it is natural to first set λ to be large; then, to make it smaller, step-by-step. In general, let $\nu=2$, when $\Delta Y<0$, then let λ be smaller as $\lambda^{new}=\lambda^{old}/\nu$ and continue the calculation. On the other hand, when $\Delta Y\geqq 0$, let λ be larger as $\lambda^{new}=\lambda^{old}*\nu$ and again try the same iteration term of calculation.

(3) Scaling of parameters

Though a scaling of the parameters does not affect the convergence while using NEWTON's method, it affects the convergence while using the steepest descent method. The reason for this phenomenon is that the scaling is equivalent to a simple linear transformation, and does not maintain orthogonality. MARQUARDT's method is similar to the steepest descent method when λ is at first large; thus, it is necessary to adequately scale the parameters. The scaling of the parameters can be expressed as follows:

$$\theta_i^*=s_i\theta_i \quad , \quad \Delta\theta_i^*=s_i\Delta\theta_i \qquad (22)$$

$$\frac{\partial Y}{\partial\theta_i^*}=\frac{1}{s_i}\frac{\partial Y}{\partial\theta_i} \quad , \quad \frac{\partial^2 Y}{\partial\theta_i^*\,\partial\theta_j^*}=\frac{1}{s_is_j}\frac{\partial^2 Y}{\partial\theta_i\partial\theta_j}$$

Using matrix notation, the above can be expressed as

$$\boldsymbol{\theta}^*=\boldsymbol{S\theta} \quad , \quad \boldsymbol{\Delta\theta}^*=\boldsymbol{S\Delta\theta} \qquad (23)$$

$$\boldsymbol{g}^*=\boldsymbol{S}^{-1}\boldsymbol{g} \quad , \quad \boldsymbol{H}^*=\boldsymbol{S}^{-1}\boldsymbol{H}\boldsymbol{S}^{-1}$$

$$\boldsymbol{S}=\begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix} , \quad \boldsymbol{S}^{-1}=\begin{pmatrix} \frac{1}{s_1} & & \\ & \ddots & \\ & & \frac{1}{s_n} \end{pmatrix}$$

$\boldsymbol{S}$ is a symmetric but non-orthogonal matrix.

MARQUARDT (1963) chose $\boldsymbol{S}_1$ (for $\boldsymbol{S}$) as follows.

$$S_1=\begin{pmatrix}\sqrt{h_{11}} & & \\ & \ddots & \\ & & \sqrt{h_{nn}}\end{pmatrix}, \qquad H=(h_{ij}) \qquad (24)$$

This same type operation is also used to make a correlation matrix from a covariance matrix; then the diagonal components of $H_1^*=S_1^{-1}HS_1^{-1}$ are all 1. Therefore, it is expected that λ affects each parameter equally. Then, a good convergence is produced and the initial value of λ can be set at 0.01 for the least-squares method.

On the other hand, AKAMINE (1984, 1985) chose S_2 (for S) as follows.

$$S_2=\begin{pmatrix}\frac{1}{\theta_1} & & \\ & \ddots & \\ & & \frac{1}{\theta_n}\end{pmatrix} \qquad (25)$$

Now, it becomes easy to determine parameter errors, since each length of a parameter becomes 1. However, convergence would be a little better by S_1 rather than S_2.

(4) Partial differential of each parameter

Using MARQUARDT's method, it is necessary to calculate the partial differential of each parameter. In this program such calculations are computed directly by its expression, since convergence is slower when difference approximation is used. The partial differential expressions of each curve are given below.

For a type-0 equation, it follows from equation (2) that,

$$\frac{\partial l}{\partial l_\infty}=1-\exp h_0 \qquad (26)$$

$\theta=K,\ t_0$

$$\frac{\partial l}{\partial \theta}=-l_\infty(\exp h_0)\frac{\partial h_0}{\partial \theta}$$

$$\begin{cases}\dfrac{\partial h_0}{\partial K}=-(t-t_0)\\[2mm] \dfrac{\partial h_0}{\partial t_0}=K\end{cases}$$

For a type-1 equation, it follows from equation (7) that,

$$\frac{\partial l}{\partial l_\infty}=1-\exp h_1 \qquad (27)$$

$\theta=K,\ t_0,\ t_1,\ a$

$$\frac{\partial l}{\partial \theta}=-l_\infty(\exp h_1)\frac{\partial h_1}{\partial \theta}$$

$$\begin{cases}\dfrac{\partial h_1}{\partial K}=-\{F(t)-F(t_0)\}\\[2mm] \dfrac{\partial h_1}{\partial t_0}=K\dfrac{\partial F(t_0)}{\partial t_0}\\[2mm] \dfrac{\partial h_1}{\partial t_1}=-K\left\{\dfrac{\partial F(t)}{\partial t_1}-\dfrac{\partial F(t_0)}{\partial t_1}\right\}\\[2mm] \dfrac{\partial h_1}{\partial a}=-K\left\{\dfrac{\partial F(t)}{\partial a}-\dfrac{\partial F(t_0)}{\partial a}\right\}\end{cases}$$

$$\begin{cases} \dfrac{\partial F(t_0)}{\partial t_0}=f(t_0) \\ \dfrac{\partial F(t)}{\partial t_1}=-\dfrac{1-a}{2}\cos 2\pi(t-t_1) \\ \dfrac{\partial F(t)}{\partial a}=\dfrac{1}{2}t-\dfrac{1}{4\pi}\sin 2\pi(t-t_1) \end{cases}$$

For a type-2 equation, it follows from equation (10) that,

$$\frac{\partial l}{\partial l_\infty}=1-\exp h_2 \qquad \text{——(28)}$$

$\theta=K,\ t_0,\ t_1,\ a$

$$\frac{\partial l}{\partial \theta}=-l_\infty(\exp h_2)\frac{\partial h_2}{\partial \theta}$$

$$\begin{cases} \dfrac{\partial h_2}{\partial K}=-(t-t_0)+\dfrac{\partial \ln G(t)}{\partial K}-\dfrac{\partial \ln G(t_0)}{\partial K} \\ \dfrac{\partial h_2}{\partial t_0}=K-\dfrac{\partial \ln G(t_0)}{\partial t_0} \\ \dfrac{\partial h_2}{\partial t_1}=\dfrac{\partial \ln G(t)}{\partial t_1}-\dfrac{\partial \ln G(t_0)}{\partial t_1} \\ \dfrac{\partial h_2}{\partial a}=\dfrac{\partial \ln G(t)}{\partial a}-\dfrac{\partial \ln G(t_0)}{\partial a} \end{cases}$$

$$\frac{\partial \ln G(t)}{\partial \theta}=\frac{\partial G(t)}{\partial \theta}\Big/G(t)$$

$$\begin{cases} \dfrac{\partial G(t)}{\partial K}=\dfrac{1-a}{2}\sin\{2\theta+2\pi(t-t_1)\} \\ \qquad =\dfrac{1-a}{2}\dfrac{2\pi}{(K^2+4\pi^2)^2}\{4\pi K\cos 2\pi(t-t_1)+(K^2-4\pi^2)\sin 2\pi(t-t_1)\} \\ \dfrac{\partial G(t_0)}{\partial t_0}=-\dfrac{\partial G(t_0)}{\partial t_1} \\ \dfrac{\partial G(t)}{\partial t_1}=\dfrac{1-a}{2}\cos\theta\, 2\pi\sin\{\theta+2\pi(t-t_1)\} \\ \qquad =\dfrac{1-a}{2}\dfrac{2\pi K}{K^2+4\pi^2}\{K\sin 2\pi(t-t_1)+2\pi\cos 2\pi(t-t_1)\} \\ \dfrac{\partial G(t)}{\partial a}=\dfrac{1}{2}-\dfrac{1}{2}\cos\theta\cos\{\theta+2\pi(t-t_1)\} \end{cases}$$

The following relationships can be derived from equation (17):

$$\cos\theta\cos\{\theta+2\pi(t-t_1)\}=\frac{1}{2}[\cos\{2\theta+2\pi(t-t_1)\}+\cos 2\pi(t-t_1)]$$

$$\frac{\partial G}{\partial K}=-\frac{1-a}{2}\sin\{2\theta+2\pi(t-t_1)\}\frac{\partial \theta}{\partial K}$$

$$(\tan\theta)'=\theta'/\cos^2\theta$$

$$\frac{\partial \theta}{\partial K}=\frac{-2\pi}{K^2+4\pi^2}$$

(5) Programs of curves

Program 1 is a type-0 program. Programs 2 and 3 are parts of type-1 and type-2 programs different from a type-0 program.

The programs are based on AKAMINE (1985), but the scaling method is according to MARQUARDT (1963). GAUSS' method of elimnation is used to solve simultaneous linear equations. Because $\boldsymbol{H}$ is a symmetric matrix, only the upper triangular part of $\boldsymbol{H}$ is used for a calculation and the lower triangular part of $\boldsymbol{H}$ is used for saving the initial values of $\boldsymbol{H}$ for a further calculation, with $\lambda^{new}=\lambda^{old} * \nu$ when $\Delta Y \geqq 0$. The covergence criterion is that λ is continously made (10 times) larger. The Iteration times become large if the precision of the computer is high.

4. Consideration of models

By comparing h_1 and h_2, it can be seem that both expressions have the same form. Such a form is produced by adding a periodic changing part to a linear increasing part.

$$h_i=-K_i\{t-C_i-H_i(t)\}, \quad \text{where} \quad C_i=\text{const.} \quad \text{and} \quad H_i(t+1)=H_i(t). \qquad \text{——(29)}$$

$$\begin{cases} K_1=\dfrac{1+a}{2}K \\ C_1=\dfrac{2}{1+a}F(t_0) \\ H_1=-\dfrac{1}{2\pi}\dfrac{1-a}{1+a}\sin 2\pi(t-t_1) \end{cases} \qquad \begin{cases} K_2=K \\ C_2=t_0-\dfrac{\ln G(t_0)}{K} \\ H_2=\dfrac{\ln G(t)}{K} \end{cases}$$

Then, each type-1 and -2 curve can be surrounded on both sides by two VON BERTALANFFY curves (equation (31)) as follows:

$$\min H_i \leqq H_i \leqq \max H_i$$

$$\min t_{0i} \leqq C_i+H_i \leqq \max t_{0i} \qquad \text{——(30)}$$

$$\begin{matrix}\max \\ \min\end{matrix}\ l_i=l_\infty(1-\exp h_i^*), \quad \text{where} \quad h_i^*=-K_i(t-\begin{matrix}\min \\ \max\end{matrix}\ t_{0i}) \qquad \text{——(31)}$$

Therefore, when sampling intervals are all 1 as $t_{i+1}-t_i=1(i=1\sim n-1)$, there is only one solution of K; but t_0 has a range similar to equation (30) (type-0). One should be careful when comparing values of t_0 when using VON BERTALANFFY model (type-0).

Next, consider H_2. If $K \ll 2\pi$, the following relationships exist:

$$\cos\theta=\frac{K}{\sqrt{K^2+4\pi^2}} \fallingdotseq \frac{K}{2\pi} \ll 1, \quad \theta \fallingdotseq \frac{\pi}{2},$$

$$\text{and} \qquad \cos(\frac{\pi}{2}+t)=-\sin t.$$

If $x \ll 1$, then $\ln(1+x) \fallingdotseq x$.

Therefore the following equation is obtained from equation (17):

$$G(t)=\frac{1+a}{2}\left[1+\frac{1-a}{1+a}\cos\theta\cos\{\theta+2\pi(t-t_1)\}\right]$$

$$\ln G(t) \fallingdotseq \ln\frac{1+a}{2}-\frac{1-a}{1+a}\frac{K}{2\pi}\sin 2\pi(t-t_1).$$

The first term on the right side would be eliminated by the term $-\ln G(t_0)$;

then, it becomes

$$H_2 \doteqdot -\frac{1}{2\pi}\frac{1-a}{1+a}\sin 2\pi(t-t_1)=H_1.$$

This expression can be regarded as

$$h_1 \doteqdot h_2. \qquad (32)$$

In general, K for aquatic organisms is much smaller than 2π ($K \ll 2\pi$). Also, an expansion of a type-1 equation is more natural than that of a type-2 equation. Therefore, it is practical to use only type-1.

5. Extreme points of the growth rate

Though the water temperature is extreme at $f'=0$, the growth rate is extreme at $l''=0$. Both points differ as follows:

$$\frac{d^2l}{dt^2}=-l_\infty \exp h_i(h_i'^2+h_i'')=0$$

$$h_i'^2+h_i''=0. \qquad (33)$$

The above becomes the following for a type-1 situation:

$$h_i'=-Kf, \qquad h_i''=-Kf'$$

and $Kf^2-f'=0.$ (34)

This equation can not be solved analytically and NEWTON's method should be used :

$$y=Kf^2-f', \qquad y'=2Kff'-f''$$

$$\Delta t=-\frac{y}{y'} \qquad (35)$$

This iteration converges easily since the number of parameters is only 1.

In a type-2 equation, from equations (10) and (33) with $G'-KG=-Kf$, it becomes

$$Kf-f'=0. \qquad (36)$$

However, in this case equation (36) is more easily obtained directly from equation (5). Though this can be solved analytically, NEWTON's method has also been used, just as for a type-1 situation.

6. Error of parameters

An estimation of parameter errors is performed according to AKAMINE (1985). The following approximate equation is considered to be in the neighbourhood of the solution :

$$\Delta Y \doteqdot \frac{1}{2}{}^t\boldsymbol{\Delta\theta}\boldsymbol{H}\boldsymbol{\Delta\theta}. \qquad (37)$$

This equation shows that

$\boldsymbol{V}=\langle\boldsymbol{\Delta\theta}^t\boldsymbol{\Delta\theta}\rangle\sim\boldsymbol{H}^{-1}$, where $\langle\ \rangle$: expected value. (38)

The most basic method for an estimation is to move only one parameter with the other parameters fixed. Then, from equation (37), ΔY becomes

$$\Delta Y=\frac{1}{2}h_{ii}(\Delta\theta_i)^2.$$

Therefore, the parameter which has a small diagonal component of $\boldsymbol{H}$ seems to be changeable. This is equivalent to saying that the parameter which has a large diagonal component of $\boldsymbol{H}^{-1}$ is changeable from equation (38).

Next, it is easy to consider the relationship among the parameters of the correlation matrix ($\boldsymbol{R}$). $\boldsymbol{R}$ is obtained, as follows, from equation (38).

$$\boldsymbol{R}=\left(\frac{h_{ij}^{-1}}{\sqrt{h_{ii}^{-1}h_{jj}^{-1}}}\right)=\boldsymbol{S}_r\boldsymbol{H}^{-1}\boldsymbol{S}_r \qquad \text{—(39)}$$

$$\boldsymbol{H}^{-1}=(h_{ij}^{-1}),\quad \boldsymbol{S}_r=\begin{pmatrix}\frac{1}{\sqrt{h_{11}^{-1}}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{h_{nn}^{-1}}}\end{pmatrix}$$

Next, consider the extreme points of the following L^2 when $\varDelta Y=$ const.,

$$L^2={}^t\boldsymbol{\varDelta\theta}\boldsymbol{\varDelta\theta}=\varDelta\theta_1^2+\cdots\cdots\cdots+\varDelta\theta_n^2.$$

This becomes

$$\boldsymbol{\varDelta\theta}=k\boldsymbol{e}_i \qquad \text{—(40)}$$

$$\boldsymbol{H}^{-1}\boldsymbol{e}_i=\lambda_i\boldsymbol{e}_i,\quad (\boldsymbol{H}\boldsymbol{e}_i=\frac{1}{\lambda_i}\boldsymbol{e}_i),\quad \text{where}\quad {}^t\boldsymbol{e}_i\boldsymbol{e}_j=\begin{cases}0 & (i\neq j)\\ 1 & (i=j)\end{cases}$$

Therefore, these are the eigenvalues and eigenvectors of $\boldsymbol{H}^{-1}$. For these vectors, the approximate equation of $\varDelta Y$ becomes

$$\varDelta Y=\frac{1}{2}\frac{k^2}{\lambda_i}. \qquad \text{—(41)}$$

For a test, the following equation was used (Draper and Smith 1966) :

$$\frac{\varDelta Y}{Y_0}=\frac{p}{m-p}F(p,\ m-p,\ 1-\alpha) \qquad \text{—(42)}$$

m : number of samples
p : number of parameters
α : confidence level

Equation (42) has the correct relationship for linear models; however, this is a non-linear model and equation (42) can only be used as an approximation.

Next, consider the influences on the scaling of the parameters. Using equation (23), equation (37) becomes

$$\varDelta Y\doteqdot\frac{1}{2}{}^t\boldsymbol{\varDelta\theta}\boldsymbol{S}\boldsymbol{S}^{-1}\boldsymbol{H}\boldsymbol{S}^{-1}\boldsymbol{S}\boldsymbol{\varDelta\theta}=\frac{1}{2}{}^t\boldsymbol{\varDelta\theta}^*\boldsymbol{H}^*\boldsymbol{\varDelta\theta}^*.$$

And equation (40) becomes

$$\boldsymbol{H}^*\boldsymbol{e}_i^*={}^t\boldsymbol{S}^{-1}\boldsymbol{H}\boldsymbol{S}^{-1}\boldsymbol{S}\boldsymbol{e}_i={}^t\boldsymbol{S}^{-1}\frac{1}{\lambda_i}\boldsymbol{e}_i$$

$$\boldsymbol{H}^*\boldsymbol{e}_i^*={}^t\boldsymbol{S}^{-1}\boldsymbol{S}^{-1}\frac{1}{\lambda_i}\boldsymbol{e}_i^*.$$

Therefore, $\boldsymbol{e}_i^*$ is not an eigenvector of $\boldsymbol{H}^*$, since $\boldsymbol{S}$ is not an orthogonal matrix (${}^t\boldsymbol{S}=\boldsymbol{S}\neq\boldsymbol{S}^{-1}$). This is easy to understand from the following relationship :

$$L^{2*}={}^t\Delta\boldsymbol{\theta}^*\Delta\boldsymbol{\theta}^*={}^t\Delta\boldsymbol{\theta}S^2\Delta\boldsymbol{\theta}=s_1^2\Delta\theta_1^2+\cdots\cdots\cdots\cdots+s_n^2\Delta\theta_n^2.$$

In general matrix theory, the eigenvalue resolution is as follows :

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}.$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are "simillar" as defined by the relation

$$\boldsymbol{B}=\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}.$$

It follows that

$$\boldsymbol{P}^{-1}(\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{-1})\boldsymbol{P}=(\boldsymbol{Q}^{-1}\boldsymbol{P})^{-1}\boldsymbol{B}(\boldsymbol{Q}^{-1}\boldsymbol{P})=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}.$$

Then, $\boldsymbol{B}$ has the same eigenvalues as $\boldsymbol{A}$ and its eigenvectors are

$$\boldsymbol{e}_i^*=\boldsymbol{Q}^{-1}\boldsymbol{e}_i.$$

However, because ${}^t\boldsymbol{S} \neq \boldsymbol{S}^{-1}$, $\boldsymbol{H}^*$ and $\boldsymbol{H}$ are not "similar". Thus, such relationships do not exist between $\boldsymbol{H}^*$ and $\boldsymbol{H}$.

From the above it can be seem that the choice of the scalling method is an important problem for the estimation of parameter errors on eigenvectors of $\boldsymbol{H}$. In this paper, $\boldsymbol{S}_2$ is chosen to be the same as that of Akamine (1985). It is therefore possible to treat parameter errors as a ratio of error to its own parameter length. In a practical calculation, first set the solution values of the parameters as the initial values for the program, then, run and stop at line 135. Finally, output values of HESSIAN (I, J). Because these values are $\boldsymbol{H}_1^*=\boldsymbol{S}_1^{-1}\boldsymbol{H}\boldsymbol{S}_1^{-1}$, the operation $\boldsymbol{H}_2^*=\boldsymbol{S}_2^{-1}\boldsymbol{S}_1\boldsymbol{H}_1^*\boldsymbol{S}_1\boldsymbol{S}_2^{-1}$ is necessary to obtain $\boldsymbol{H}_2^*$. In practice, it is sufficient to calculate the expression as follows.

HESSIAN (I, J)*SCALE (I)*SCALE (J)*P (I)*P (J)

Next, consider the correlation matrix ($\boldsymbol{R}$). It becomes as follows from equation (39) :

$$\boldsymbol{R}^*=\boldsymbol{S}_r^*\boldsymbol{H}^{*-1}\boldsymbol{S}_r^*$$

$$\boldsymbol{H}^{*-1}=(\boldsymbol{S}^{-1}\boldsymbol{H}\boldsymbol{S}^{-1})^{-1}=\boldsymbol{S}\boldsymbol{H}^{-1}\boldsymbol{S}$$

$$\boldsymbol{S}_r^*=\begin{pmatrix}\frac{1}{\sqrt{h^{*-1}_{11}}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{h^{*-1}_{nn}}}\end{pmatrix}=\begin{pmatrix}\frac{1}{s_1\sqrt{h^{-1}_{11}}} & & \\ & \ddots & \\ & & \frac{1}{s_n\sqrt{h^{-1}_{nn}}}\end{pmatrix}$$

Then it becomes

$$\begin{array}{ccc}\boldsymbol{H}^{-1} & \xrightarrow{\quad\boldsymbol{S}\quad} & \boldsymbol{H}^{*-1}\\ \boldsymbol{S}_r\downarrow & & \downarrow\boldsymbol{S}_r^*\\ \boldsymbol{R} & \xrightarrow[\boldsymbol{S}_r^*\,\boldsymbol{S}\boldsymbol{S}_r^{-1}]{} & \boldsymbol{R}^*\end{array}$$

$$S_r^* SS_r^{-1}=I \qquad (S_r^* S=S_r)$$

Obviously, this can be expressed as

$$R^*=R.$$

Therefore, a correlation matrix is never affected by the scaling of parameters. This is obvious from the definition of the correlation coefficient.

The correlation matrix is regarded as the covariance matrix of the parameters standardized according to their standard deviation. However, S_r is regarded as one of the scalings of the parameters. If $H^{-1}(V)$ is a diagonal matrix then $S_1=S_r$. This means that paraneters are independent of each other. Therefore, this relationship does not exsist in general.

One of the typical analysing method using eigenvalues and eigenvectors is the principal component analysis. In this method, the user chooses either a covariance matrix or a correlation matrix for his object. However, in general, it is better to choose a correlation matrix.

In this paper $H_2^{*-1}=S_2H^{-1}S_2$ is used. If a part of the solutions of parameters is near 0 (for example. $t_1=0$ or $a=0$), it is not sufficient since the part of errors (t_1, a) is evaluated too large. In general, though it is better to use a correlation matrix, there is another method that the user sets s_i for each parameter. In such cases, since the calculations are all the same, calculation details are omitted.

7. An example computation

(1) The data for computation

The artificial data in Table 1 were used for a test computation. This periodically oscillating data is set as $l_\infty=100$, $K=0.5$ and $t_0=0.5$.

Table 1. The artificial data for the test of Program 1~3.

i	t_i	l_{0i}	σ_i	i	t_i	l_{0i}	σ_i	i	t_i	l_{0i}	σ_i
1	0.5	5	3	8	2.0	47	2	15	3.5	80	3
2	0.8	12	3	9	2.2	54	3	16	4.0	82	2
3	1.0	18	2	10	2.4	63	3	17	4.5	87	3
4	1.2	30	4	11	2.5	66	3	18	5.0	88	3
5	1.3	36	3	12	2.8	69	3	19	7.5	99	5
6	1.5	42	3	13	3.0	68	6	20	10.0	99	2
7	1.7	45	3	14	3.2	74	3				

(2) Results of computations

The results of computations are given in Table 2. The graphs of the results were drawn using an XY-plotter (Fig. 1). A graph of a type-2 equation is omitted because it is the same as that of a type-1 equation and they are difficult to distinguish. This is because equation (32) exists, approximately, for $K \ll 2\pi$.

Table 2. Results of the computation by Program 1~3 for the data in Table 1.

	Times of iterations	l_∞	K $(K_1)^{1)}$	t_0	t_1	a	Y_0
Initial value	0	100	0.5	0.5	0.25	0.0	
type-0	3	100.916	0.478494	0.495538			19.9981
type-1	5	100.623	0.870266 (0.487011)	0.388283	0.229144	0.119223	3.93503
type-2	6	100.615	0.487133	0.388253	0.21616	0.119418	3.98904

1) $K_1 = \frac{1+a}{2} K$

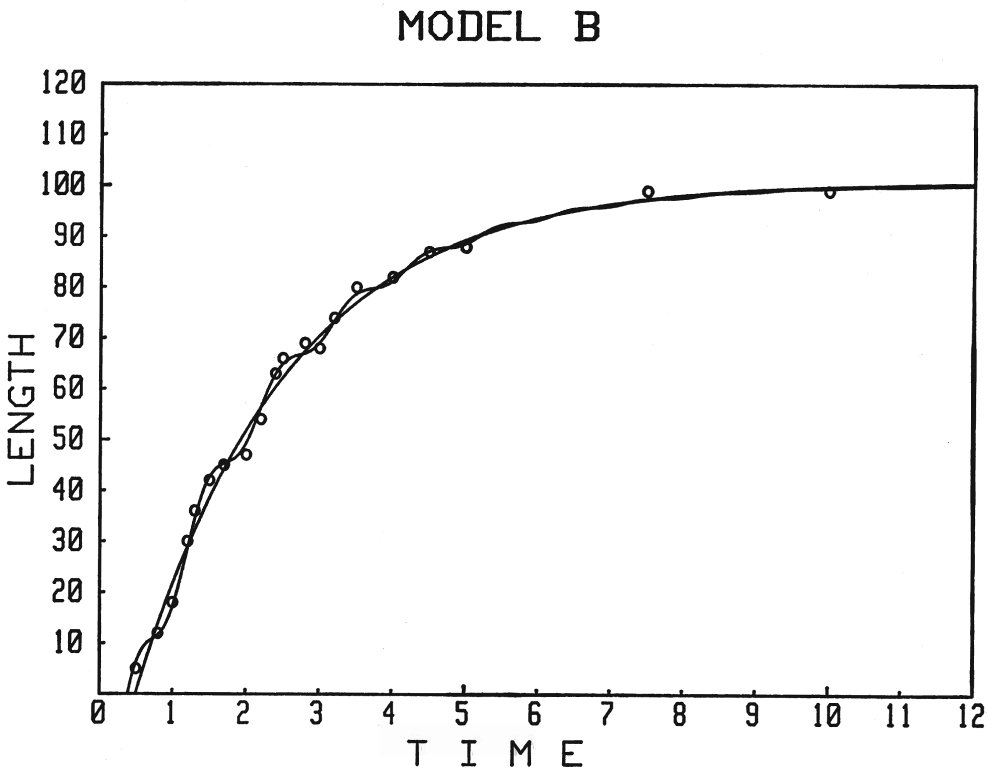

Fig. 1. Graphs of type-0 and type-1 for von Bertalanffy model in Table 2. (The periodically oscillating curve is type-1 and the other is type-0.)

The results of calculations for maxl_i, minl_i equation (31) are listed in Table 3 and Fig. 2. Also, in this case a graph of a typh-2 equation is omitted because it is the same as that of a type-1 equation. The values of t for extreme points of the growth rate are given in Table 4. These values are rather different, but they seem natural since the values of t_1 are even more different than the other parametes in Table 2.

Table 3. Min t_0 and max t_0 for each type.

	min t_0	t_0	max t_0
type-1	0.368421	0.388283	0.618916
type-2	0.367569	0.388253	0.617524

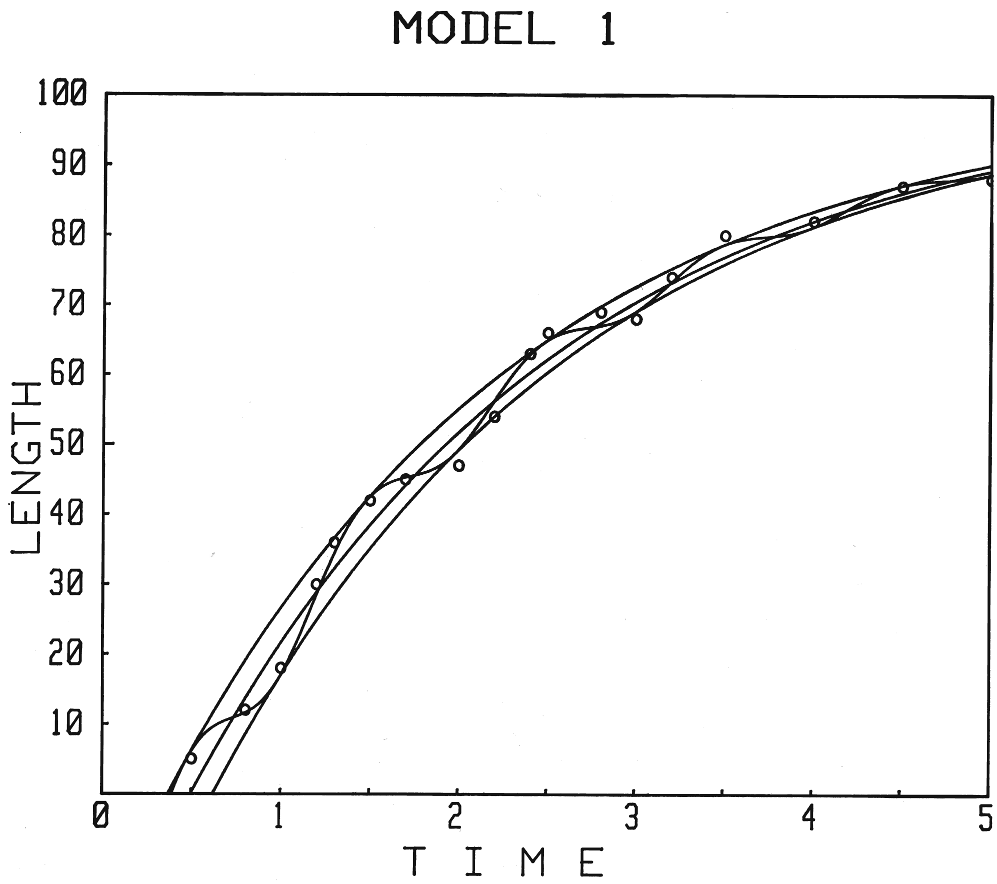

Fig. 2. Graphs of type-1 and type-0 for min t_0, t_0 and max t_0 in Table 3. (The upper curve is type-0 for min t_0, the lower curve is type-0 for max t_0, and the middle curves are type-0, 1 in Fig. 1.)

Table 4. Extreme points for each type.

	maximal point	t_1	minimal point
type-1	0.180361	0.229144	0.729856
type-2	0.188181	0.21616	0.71951

Thus, it has been proved that the type-2 equation results in the same type curves as for the type-1 equation. Therefore, most of the following descriptions are according to type-1 only.

(3) Estimation of parameter errors

The results of an estimation of parameter errors are listed in Tables 5 and 6 and their graphs are given in Figs. 3 and 4. The eigenvectors of the type-1 equation show that a is the most changeable. This is because intervals of data are clearly too large (Fig. 4). Even von Bertalanffy model (type-0) may be sufficient for this data.

BASIC programs (Householder transform, bisection method, Wielandt's inverse iteration) of 玄・井田(1983) were used to compute an inverse matrix and the eigenvalues and eigenvectors.

Table 5-a. Results of the calculation to estimate errors of each parameter (type-0).

parameter	l_∞	K	t_0	$\lambda_i \times 10^3$ (%)
solution (s_2)	100.916	0.478494	0.495538	
s_1	1.02521	95.6023	39.4126	
$g_0 \times 10^4$	- 3.44588	- 2.8637	- 1.70757	
H^*_1	1	0.847855	- 0.512361	
		1	- 0.735915	
			1	
H^*_2	10704	4012.72	- 1035.29	
		2092.62	- 657.483	
			381.438	
$R^*_2 \setminus V^*_2$ 1)	0.367830	- 0.854352	- 0.474288	
	-.81	3.02680	2.89841	
	-.31	.66	6.33034	
e_1	0.108342	- 0.506386	- 0.855474	8.10607 (83.35)
e_2	0.363887	- 0.780615	0.508160	1.53827 (15.82)
e_3	- 0.925160	- 0.366357	- 0.099309	0.08060 (0.83)

1) $V^*_2 = H^{*}_2{}^{-1}$

Table 5-b. Results of the calculation to estimate errors of each parameter (type-1).

parameter	l_∞	K	t_0	t_1	a	$\lambda_i \times 10^3$ (%)
solution (s_2)	100.623	0.870266	0.388283	0.229144	0.119223	
s_1	1.02897	52.0887	56.9683	24.8581	48.9777	
$\boldsymbol{g}_0 \times 10^4$	-4.92649	-3.23386	3.35114	-0.96926	-5.43809	
$\boldsymbol{H}^*_1$	1	0.847876	-0.506777	0.278961	0.802523	
		1	-0.734091	0.405565	0.975324	
			1	-0.681135	-0.815085	
				1	0.474802	
					1	
$\boldsymbol{H}^*_2$	10720.1	3979.49	-1160.64	164.52	485.195	
		2054.9	-736.084	104.721	258.168	
			489.287	-85.8206	-105.279	
				32.4454	15.7924	
					34.097	
$\boldsymbol{R}^*_2 \backslash \boldsymbol{V}^*_2$ 1)	0.372833	-0.8737	-0.5341	-0.4108	-0.15	
	-.37	14.9671	3.3636	1.7924	-112.11	
	-.26	.26	11.5398	13.7500	62.33	
	-.09	.06	.52	60.1433	6.87	
	-.01	-.89	.56	.03	1069.54	
$\boldsymbol{e}_1$	-0.000083	-0.104191	0.058067	0.007254	0.992834	1083.51 (93.81)
$\boldsymbol{e}_2$	0.009028	-0.054637	-0.235281	-0.970231	0.015118	63.4753 (5.50)
$\boldsymbol{e}_3$	0.134000	-0.586609	-0.767557	0.220129	-0.018265	6.6023 (0.57)
$\boldsymbol{e}_4$	-0.358517	-0.714410	-0.582644	0.099412	0.108291	1.4048 (0.12)
$\boldsymbol{e}_5$	0.923814	0.362865	-0.112477	0.016133	0.044554	0.0803 (0.00)

1) $\boldsymbol{V}^*_2 = \boldsymbol{H}^{*}_2{}^{-1}$

Table 6-a The approximate value of ΔY 1) to estimate the confidence interval.

	p	m	α (%)	F	Y_0	ΔY 1)
type-0	3	20	5	3.197	19.9981	11.28
			1	5.185		18.30
type-1	5	20	5	2.901	3.93503	3.805
			1	4.556		5.976

1) $\dfrac{\Delta Y}{Y_0} = \dfrac{p}{m-p} F(p, m-p, 1-\alpha)$

Table 6-b. The confidence interval of each parameter on $\boldsymbol{e}_1$ in Table 5.

	k[1)]	l_∞	K	t_0	t_1	a	ΔY[2)]
	-0.33	97.308	0.558454	0.635432			17.70
	-0.27	97.964	0.543916	0.609996			11.24
type-0	0.00	100.916	0.478494	0.495538			0.00
	0.30	104.196	0.405803	0.368362			10.86
	0.38	105.071	0.386419	0.334449			17.95
	-1.7	100.637	1.02441	0.349954	0.226318	-0.082004	5.842
	-1.4	100.635	0.997209	0.356718	0.226817	-0.046493	3.353
type-1	0.0	100.623	0.870266	0.388283	0.229144	0.119223	0.000
	1.6	100.610	0.725188	0.424357	0.231804	0.308613	3.446
	1.9	100.607	0.697986	0.431121	0.232302	0.344124	5.764

1) $\Delta\theta = k\boldsymbol{e}_1$
2) $\Delta Y = Y(\theta_0 + \Delta\theta) - Y(\theta_0)$

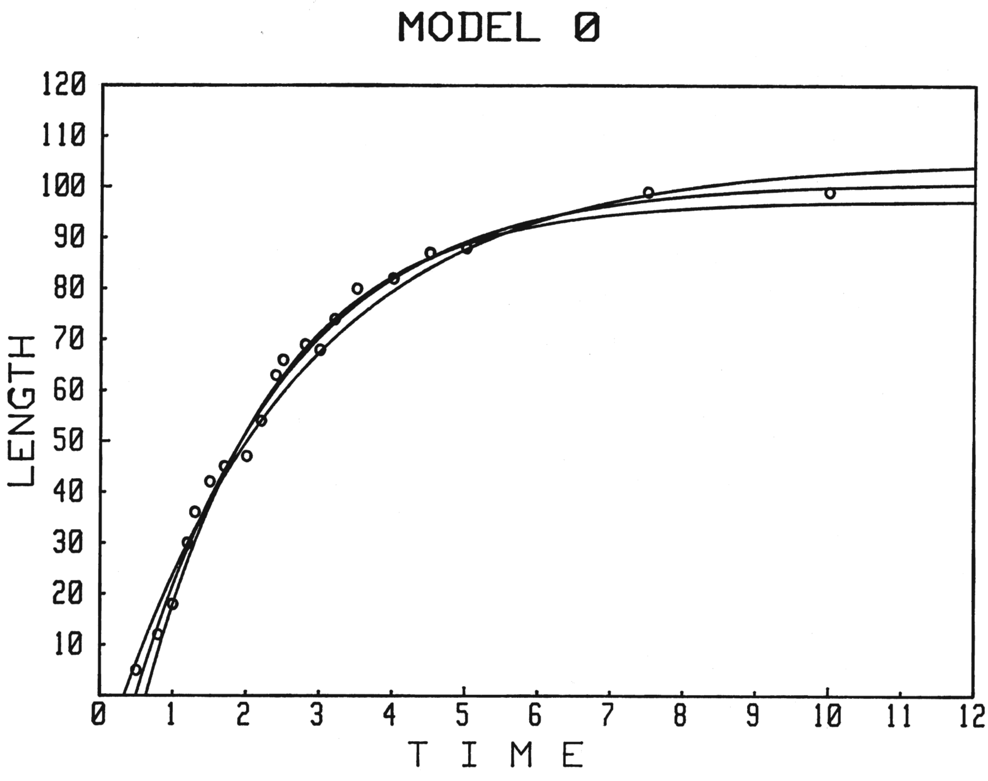

Fig. 3. Graphs of type-0 in Table 6-b. (The steepest curve is that for $k=-0.33$, the most gentle curve is that for $k=0.38$, and the middle curve is that for $k=0.00$.)

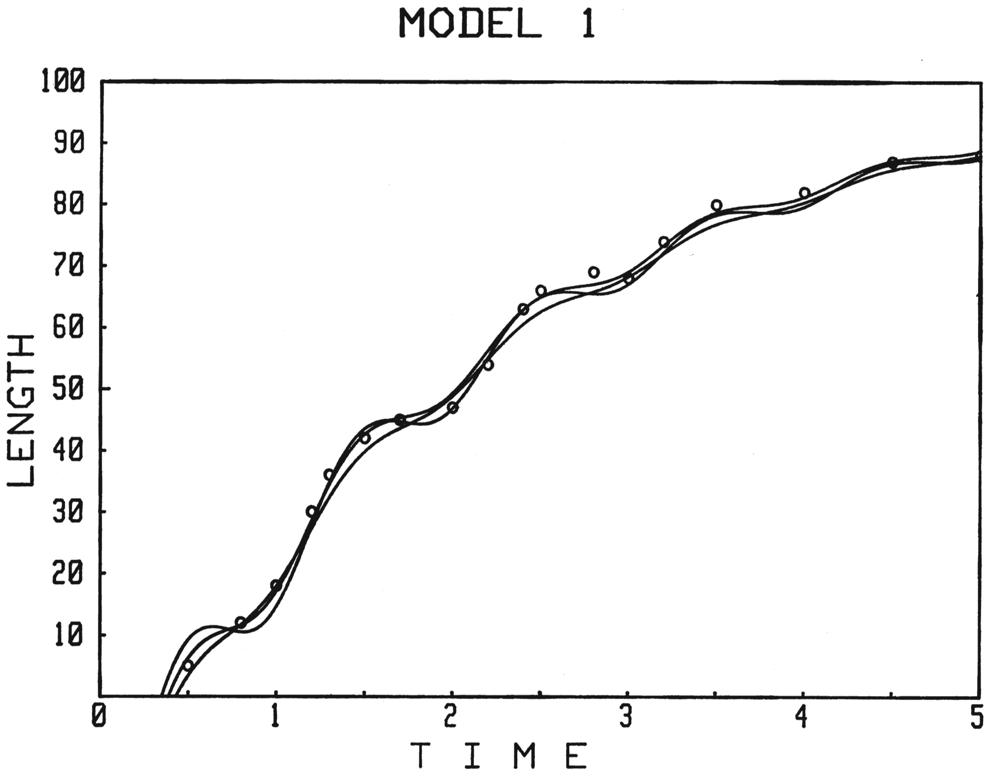

Fig. 4. Graphs of type-1 in Table 6-b. (The most oscillating curve is that for $k=-1.7$, the most gentle curve is that for $k=1.9$, and the middle curve is that for $k=0.0$.)

(4) Comparision with WALFORD's graph

Results obtained by using WALFORD's graph are compared. WALFORD's graph is described as follows from equation(2) :

$$l_{t+1}=al_t+l_\infty(1-a),\quad \text{where} \quad a=\exp(-K) \qquad \text{——(43)}$$

$$\text{and} \quad t_0=t+\frac{1}{K}\ln(1-\frac{l_t}{l_\infty}). \qquad \text{——(44)}$$

K and l_∞ are calculated using the regression line of equation (43), and t_0 is calculated for each t by equation (44). Results are given in Table 7 and Figs. 5 and 6. This method is practical enough for this data, since it is within a 99% confidence interval for Tables 5 and 6. However, it is regrettable that this method cannot use anything except regular intervallic data and, therefore, cannot draw enough information from the data in Fig. 6.

Table 7. The data and results of WALFORD's graph.

t_i	l_{0i}	l_{0i+1}	t_0 1)
1.0	18	47	.556
2.0	47	68	.595
3.0	68	82	.521
4.0	82	88	.371
5.0	(88)		.627
$y=.652x+36.2$ ($l_\infty=104$, $K=.428$)	$\bar{t}_0=.534$		$\Delta Y=18.0441$ ($Y_0=38.0422$)

1) $t_0=t_i+\frac{1}{K}\ln(1-\frac{l_{0i}}{l_\infty})$

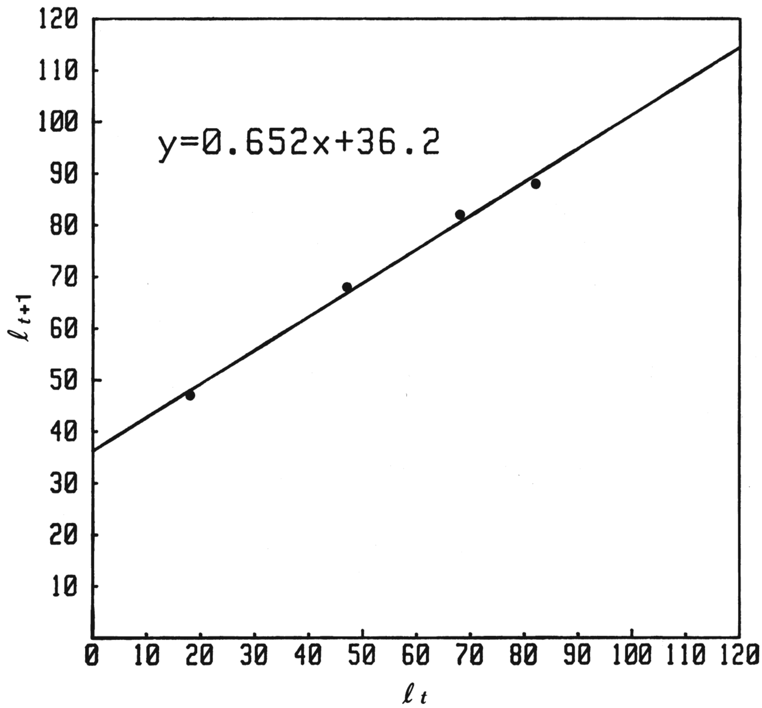

Fig. 5. WALFORD' s graph for the data in Table 7.

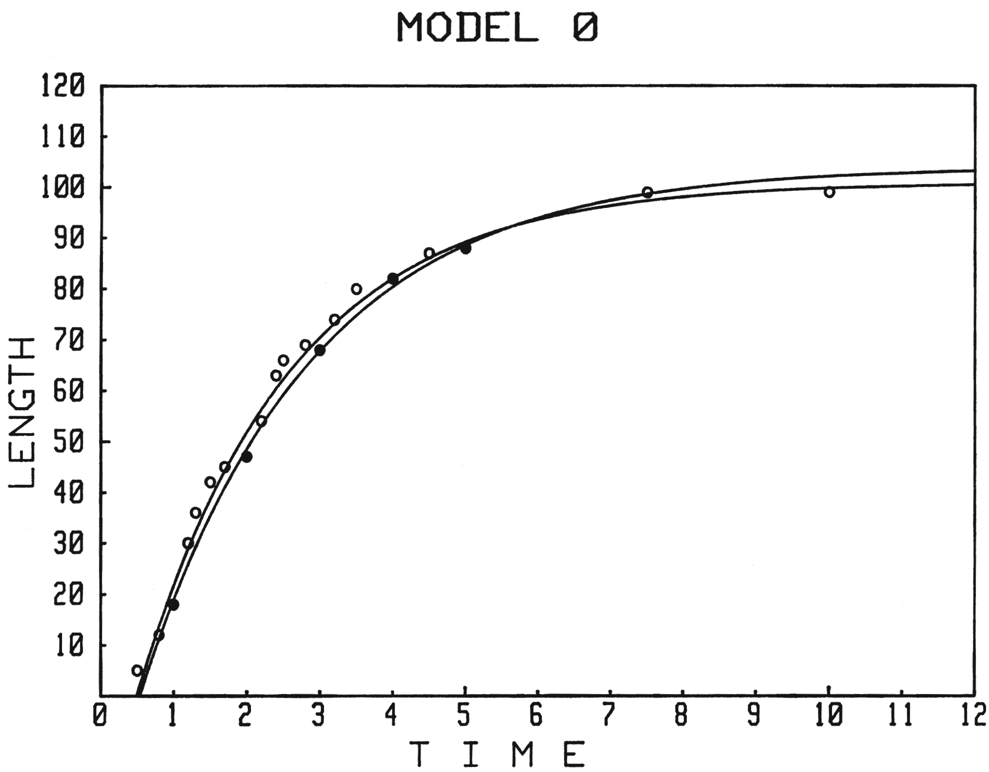

Fig. 6. Graphs of type-0 in Table 2 and Table 7. (Only black circles are used for WALFORD's graph.)

(5) The computing time

These programs were developed while aiming at an easily understood algorithm and an ability to use it with other curves. Thus, the computing efficiency could not to be so good. For example, all expressions are computed by a DEFFN statement. Then, the same computation is performed many times. In the case of using PC-9801F (NEC), each program required less than 5 minutes. Therefore, these are sufficient for practical use. When using a slow computer, it is better to impose the computations of all expressions into the main program, just like AKAMINE(1985).

The number of iterations seems to be larger in the case of bad initial values or a high precision computer. Also, it is natural that the computing time becomes longer in the case of a large data entry.

III. Expansion of logistic and GOMPERTZ models

1. Expansion of logistic model

This expansion is the same as that for VON BERTALANFFY model. The differential equation of this model is

$$\frac{dl}{dt}=al(l_\infty-l)\ . \qquad (45)$$

The integral of equation (45) with the initial condition, when $t=t_0$ let $l=l_\infty/2$, is

$$l=\frac{l_\infty}{1+\exp h_0}\ . \qquad (46)$$

This is a "type-0" equation. The differential of equation (46) is

$$\frac{dl}{dt}=\frac{a^*\exp h_0}{\{1+\exp h_0\}^2}\ . \qquad (47)$$

The "type-1" model for the growth rate is

$$\frac{dl}{dt}=al(l_\infty-l)f(t)\ . \qquad (48)$$

The integral of equation (48) is

$$l=\frac{l_\infty}{1+\exp h_1}\ . \qquad (49)$$

From equation (46), the above can be transformed as

$$\frac{1}{l}=\frac{1}{l_\infty}(1+\exp h_0)\ ,\qquad \frac{d}{dt}\left(\frac{1}{l}\right)=a^+\exp h_0 \qquad \text{where} \qquad a^+=-\frac{K}{l_\infty}\ .$$

Then, the "type-2" model for the growth rate is

$$\frac{d}{dt}\left(\frac{1}{l}\right)=a^+(\exp h_0)f(t)\ . \qquad (50)$$

The integral of equation (50) is ;

$$l=\frac{l_\infty}{1+\exp h_2}\ . \qquad (51)$$

It seems that the type-2 expansion is not so natural as that of type-1.

The partial differential of each parameter is as follows :

$$\frac{\partial l}{\partial l_\infty}=\frac{1}{1+\exp h_i} \qquad (52)$$

$\theta=K,\ t_0,\ t_1,\ a$

$$\frac{\partial l}{\partial\theta}=-l_\infty\frac{\exp h_i}{(1+\exp h_i)^2}\frac{\partial h_i}{\partial\theta}$$

The extreme points of the growth rate are

$$h'^2_i(1-\exp h_i)+h'_i{}'(1+\exp h_i)=0\ . \qquad (53)$$

In a type-1 equation, it is

$$Kf^2(1-\exp h_1)-f'(1+\exp h_1)=0\ . \qquad (54)$$

2. Expansion of GOMPERTZ model

This expansion is also the same as that of the former models. The differential equation of this model is

$$\frac{dl}{dt}=al(\ln l_\infty-\ln l)\ . \qquad (55)$$

The integral of equation (55) with the initial condition, when $t=t_0$ let $l=l_\infty/e$, is

$$l=l_\infty\exp(-\exp h_0)\ . \qquad (56)$$

This is a "type-0" equation. The differential of equation (56) is

$$\frac{dl}{dt}=a^*\exp(-\exp h_0)\exp h_0\ . \qquad (57)$$

The "type-1" model for the growth rate is

$$\frac{dl}{dt}=al(\ln l_\infty-\ln l)f(t)\ . \qquad (58)$$

The integral of equation (58) is

$$l=l_\infty\exp(-\exp h_1)\ . \qquad (59)$$

From equation (56), the above can be transformed as

$$\ln l=\ln l_\infty-\exp h_0\ , \qquad \frac{d(\ln l)}{dt}=\frac{1}{l}\frac{dl}{dt}=a^{\circ}\exp h_0 \qquad \text{where} \qquad a^{\circ}=K$$

Then, the "type-2" model for the growth rate is

$$\frac{d(\ln l)}{dt}=a^{\circ}(\exp h_0)f(t)\ . \qquad (60)$$

The integral of equation (60) is

$$l=l_\infty\exp(-\exp h_2)\ . \qquad (61)$$

It also seems that the type-2 expansion is not so natural as that of type-1.

The partial differential of each parameter is as follows :

$$\frac{\partial l}{\partial l_\infty}=\exp(-\exp h_i) \qquad (62)$$

$$\theta=K,\ t_0,\ t_1,\ a$$

$$\frac{\partial l}{\partial \theta}=-l_\infty\exp(-\exp h_i)\exp h_i\frac{\partial h_i}{\partial \theta}$$

The extreme points of growth rate are

$$h'^2_i(1-\exp h_i)+h'_i{}'=0\ . \qquad (63)$$

In a type-1 equation, it is

$$Kf^2(1-\exp h_1)-f'=0\ . \qquad (64)$$

3. Programs and results

Program 4 and 5 are the parts of logistic and GOMPERTZ model programs different from VON BERTALANFFY model program. These programs were tested using artificial data. These data are listed in Tables 8 and 9 and the results of the computations are given in Table 10 and Figs. 7 and 8. Computations for type-2 and other calculations has been omitted.

Table 8. The artificial data for the test of program 4.

i	t_i	l_{0i}	σ_i	i	t_i	l_{0i}	σ_i	i	t_i	l_{0i}	σ_i
1	- 4.0	1	2	10	- 0.7	38	4	19	2.0	87	3
2	- 3.5	3	2	11	- 0.3	39	5	20	2.3	92	4
3	- 3.0	4	3	12	0.0	48	4	21	2.8	93	3
4	- 2.8	7	3	13	0.3	61	4	22	3.0	96	3
5	- 2.4	7	3	14	0.6	62	4	23	3.5	97	2
6	- 2.0	13	4	15	0.8	66	5	24	4.0	98	3
7	- 1.7	17	3	16	1.0	75	4	25	5.0	99	3
8	- 1.4	18	3	17	1.3	81	3				
9	- 1.0	25	4	18	1.7	82	3				

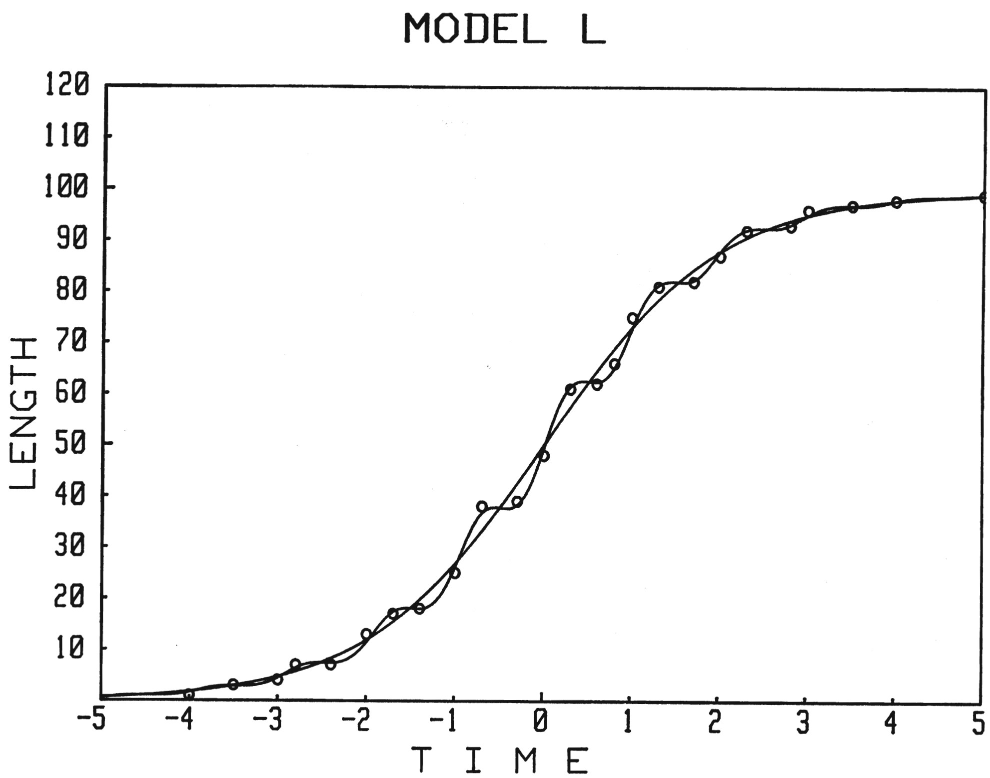

Fig. 7. Graphs of type-0 and type-1 for logistic model in Table 10. (The periodically oscillating curve is type-1 and the other is type-0.)

Table 9. The artificial data for the test of program 5.

i	t_i	l_{0i}	σ_i	i	t_i	l_{0i}	σ_i	i	t_i	l_{0i}	σ_i
1	- 1.3	1	3	8	0.2	50	2	15	2.0	88	2
2	- 1.0	8	3	9	0.4	55	3	16	2.3	92	3
3	- 0.8	16	2	10	0.7	56	5	17	2.7	92	3
4	- 0.5	19	3	11	1.0	67	3	18	3.0	94	3
5	- 0.3	24	3	12	1.2	77	3	19	4.0	98	2
6	- 0.1	29	4	13	1.5	81	2	20	5.0	98	3
7	0.0	40	4	14	1.8	82	3				

Table 10. Reaults of computation by program 4, 5 for the data in Table 8, 9.

		Times of iterations	l_∞	K	t_0	t_1	a	Y_0
	Initial value	0	100	1.0	0.0	0.0	0.0	
logistic	type-0	5	99.7154	1.00196	-0.003079			7.28418
	type-1	7	99.7416	2.13245	0.010413	0.014517	-0.050024	1.27517
GOMPERTZ	type-0	4	99.1726	0.993326	-0.080152			17.8405
	type-1	9	99.7023	1.94285	-0.014149	0.006776	0.015631	0.986610

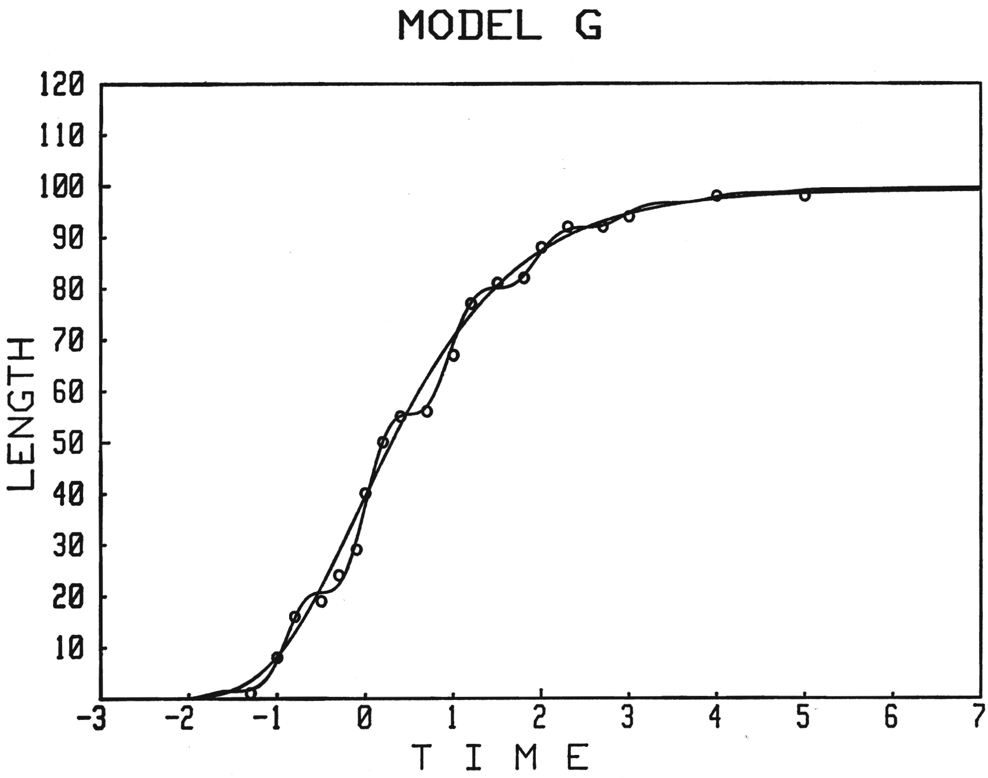

Fig. 8. Graphs of type-0 and type-1 for GOMPERTZ model in Table 10. (The periodically oscillating curve is type-1 and the other is type-0.)

IV. Conclusion

Two expansions, type-1 and type-2 were considered for each growth model. Type-2 converged for the same curves as type-1. Type-1 is easier to expand and treat than type-2. Thus, it is sufficient to use only type-1 as the expansion model.

Although it is possible to use a more complex expression for f, it becomes more difficult to treat for programing and to understand the relationship of each parameter. This expansion model seems sufficient for expressing the growth characteristics using only a few parameters.

Though these programs are not very good regarding computing efficiency, they are sufficiently practical and make it easy to understand algorithm and to apply them to other curves.

Acknowledgements

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```
10      REM
20      REM   BERTALANFFY - MARQUARDT
30      REM
100     REM  MAIN ROUTINE
110     GOSUB *VARIDEF
120     GOSUB *DATAREAD
130     GOSUB *INITIAL
140     FOR IREP=1 TO NIT
150        GOSUB *SUMUP
160        GOSUB *CALEQAT
170     NEXT IREP
180     PRINT "ITERATION WAS COMPLETED."
190     PRINT
200     IREP=IREP-1
210     GOSUB *PRINTOUT
220     END
300    *PEND2
310     PRINT "CONVERGENCE WAS COMPLETED."
320     PRINT
330     IREP=IREP-1
340     GOSUB *PRINTOUT
350     END
1000    *VARIDEF
1010     DEFINT I-N
1100     NP=3
1110     DEF FNEP1=EXP(-P2(2)*(TIME(K)-P2(3)))
1120     DEF FNDP1=1-FNEP1
1130     DEF FNBL =P2(1)*FNDP1
1140     DEF FNDP2=P2(1)*FNEP1*(TIME(K)-P2(3))
1150     DEF FNDP3=P2(1)*FNEP1*(-P2(2))
1800     DEF FND1 =(BLENGTH(K)-FNBL)/SIGMA(K)
1810     DEF FND2 =FND1*FND1
1820     DIM P(NP),P2(NP),PDELTA(NP)
1830     DIM DIFFER(NP),SCALE(NP),HESSIAN(NP,NP),GVECTOR(NP)
1840     RETURN
2000   *DATAREAD
2010    READ NIT,CLAMBDA,CNU
2020    PRINT "NUMBER OF ITERATION =";NIT
2030    PRINT "      LAMBDA         =";CLAMBDA
2040    PRINT "        NU           =";CNU
2050    PRINT
2060    READ N
2070    PRINT "NUMBER OF DATA =";N
2080    PRINT
2090    DIM BLENGTH(N),TIME(N),SIGMA(N)
2100    FOR I=1 TO N
2110       READ TIME(I),BLENGTH(I),SIGMA(I)
2120       PRINT "I=";I
2130       PRINT "      TIME   =";TIME(I)
2140       PRINT "      LENGTH =";BLENGTH(I)
2150       PRINT "      SIGMA  =";SIGMA(I)
2160       PRINT
2170    NEXT I
2180    RETURN
```

Program 1-a. The BASIC program for von Bertalanffy model (type-0) by Marquardt's method. (DATA : The example of the artificial data in Table 1).

```
3000  *INITIAL
3010   FOR I=1 TO NP
3020      READ P(I) : P2(I)=P(I)
3030   NEXT I
3035   STOP
3040   GOSUB *CALD2
3050   Y1=Y2
3060   IREP=0 : YDELTA=0
3070   GOSUB *PRINTOUT
3080   RETURN
4000  *SUMUP
4010   FOR I=1 TO NP
4020      GVECTOR(I)=0
4030      FOR J=I TO NP
4040      HESSIAN(I,J)=0
4050   NEXT J : NEXT I
4060   REM
4070   FOR K=1 TO N
4080      DIFFER(1)=FNDP1
4090      DIFFER(2)=FNDP2
4100      DIFFER(3)=FNDP3
4110      D1=FND1 : SS1=SIGMA(K) : SS2=SS1*SS1
4120      FOR I=1 TO NP
4130         GVECTOR(I)=GVECTOR(I)+D1*DIFFER(I)/SS1
4140         FOR J=I TO NP
4150            HESSIAN(I,J)=HESSIAN(I,J)+DIFFER(I)*DIFFER(J)/SS2
4160   NEXT J : NEXT I : NEXT K
4170   REM SCALING
4180   FOR I=1 TO NP
4190      SCALE(I)=SQR(HESSIAN(I,I))
4200   NEXT I
4210   FOR I=1 TO NP
4220      GVECTOR(I)=GVECTOR(I)/SCALE(I)
4230      FOR J=I TO NP
4240         HESSIAN(I,J)=HESSIAN(I,J)/SCALE(I)/SCALE(J)
4250   NEXT J : NEXT I
4260   REM
4270   FOR I=2 TO NP
4280      HESSIAN(I,1)=GVECTOR(I)
4290   NEXT I
4300   FOR I=2 TO NP-1
4310      FOR J=I+1 TO NP
4320         HESSIAN(J,I)=HESSIAN(I,J)
4330   NEXT J : NEXT I
4340   RETURN
5000  *CALEQAT
5010   K2=0
5020  *REPEAT
5030   K2=K2+1
5040   IF K2>11 GOTO *PEND2
5050   PRINT "K=";K2
5060   PRINT "      LAMBDA =";CLAMBDA
5070   PRINT
5080   FOR I=1 TO NP
5090      HESSIAN(I,I)=1+CLAMBDA
5100   NEXT I
```

Program 1-b. Continued.

```
5110   GOSUB *GAUSS
5120   REM SCALING
5130   FOR I=1 TO NP
5140      PDELTA(I)=PDELTA(I)/SCALE(I)
5150      P2(I)=P(I)+PDELTA(I)
5160   NEXT I
5170   REM
5180   GOSUB *CALD2
5190   IF Y2>=Y1 GOTO *PREREP
5200   REM
5210   CLAMBDA=CLAMBDA/CNU
5220   YDELTA=Y2-Y1
5230   Y1=Y2
5240   FOR I=1 TO NP
5250      P(I)=P2(I)
5260   NEXT I
5270   GOSUB *PRINTOUT
5280   RETURN
5500  *PREREP
5510   CLAMBDA=CLAMBDA*CNU
5520   FOR I=2 TO NP
5530      GVECTOR(I)=HESSIAN(I,1)
5540   NEXT I
5550   FOR I=2 TO NP-1
5560      FOR J=I+1 TO NP
5570         HESSIAN(I,J)=HESSIAN(J,I)
5580   NEXT J : NEXT I
5590   GOTO *REPEAT
6000  *CALD2
6010   Y2=0
6020   FOR K=1 TO N
6030      Y2=Y2+FND2
6040   NEXT K
6050   RETURN
7000  *GAUSS
7010   REM
7020   FOR I=1 TO NP-1
7030      FOR K=I+1 TO NP
7040         Q1=HESSIAN(I,K)/HESSIAN(I,I)
7050         GVECTOR(K)=GVECTOR(K)-Q1*GVECTOR(I)
7060         FOR J=K TO NP
7070            HESSIAN(K,J)=HESSIAN(K,J)-Q1*HESSIAN(I,J)
7080   NEXT J : NEXT K : NEXT I
7090   REM
7100   PDELTA(NP)=GVECTOR(NP)/HESSIAN(NP,NP)
7110   FOR I=NP-1 TO 1 STEP -1
7120      T1=GVECTOR(I)
7130      FOR J=I+1 TO NP
7140         T1=T1-PDELTA(J)*HESSIAN(I,J)
7150      NEXT J
7160      PDELTA(I)=T1/HESSIAN(I,I)
7170   NEXT I
7180   RETURN
```

Progam 1-c. Continued.

```
8000   *PRINTOUT
8010    PRINT "IREP=";IREP
8020    PRINT "           D2         =";Y1
8030    PRINT "           DELTA-D2   =";YDELTA
8040    PRINT "         L-INFINITY =";P(1)
8050    PRINT "             K       =";P(2)
8060    PRINT "             T0      =";P(3)
8070    PRINT
8080    RETURN
10000   DATA 50,0.01,2
10010   DATA 20
10020   DATA 0.5, 5,3,0.8,12,3,1.0,18,2,1.2,30,4,1.3,36,3
10030   DATA 1.5,42,3,1.7,45,3,2.0,47,2,2.2,54,3,2.4,63,3
10040   DATA 2.5,66,3,2.8,69,3,3.0,68,6,3.2,74,3,3.5,80,3
10050   DATA 4.0,82,2,4.5,87,3,5.0,88,3,7.5,99,5,10.0,99,2
10060   DATA 100,0.5,0.5
```

Program 1-d. Continued.

```
1020      PAI=3.14159265#
1100      NP=5
1120      DEF FNDP1=1-FNEP1
1130      DEF FNBL =P2(1)*FNDP1
1140      DEF FNDP2=FNEP2*(FNFT1(TIME(K))-FNFT1(P2(3)))
1150      DEF FNDP3=-FNEP3*FNFT3(P2(3))
1160      DEF FNDP4=FNEP3*(FNFT4(TIME(K))-FNFT4(P2(3)))
1170      DEF FNDP5=FNEP3*(FNFT5(TIME(K))-FNFT5(P2(3)))
1180      DEF   FNFT1(TM)=FNP51*TM+FNP52/2/PAI*SIN(FNTM1(TM))
1190      DEF   FNFT3(TM)=FNP51+FNP52*COS(FNTM1(TM))
1200      DEF   FNFT4(TM)=-FNP52*COS(FNTM1(TM))
1210      DEF   FNFT5(TM)=TM/2-1/4/PAI*SIN(FNTM1(TM))
1220      DEF     FNEP1=EXP(-P2(2)*(FNFT1(TIME(K))-FNFT1(P2(3))))
1230      DEF     FNEP2=P2(1)*FNEP1
1240      DEF     FNEP3=P2(2)*FNEP2
1250      DEF       FNP51=(1+P2(5))/2
1260      DEF       FNP52=(1-P2(5))/2
1270      DEF       FNTM1(TM)=2*PAI*(TM-P2(4))

4103        DIFFER(4)=FNDP4
4105        DIFFER(5)=FNDP5

8064     PRINT "            T1      =";P(4)
8067     PRINT "            A       =";P(5)

10060    DATA 100,0.5,0.5,0.25,0
```

Program 2. The different parts of the program for von Bertalanffy model (type-1) from Program 1.

```
1100    NP=5
1120    DEF FNDP1=1-FNEP2
1130    DEF FNBL =P2(1)*FNDP1
1140    DEF FNDP2=FNEP3*(-(TIME(K)-P2(3))+FNGT6(TIME(K))-FNGT6(P2(3)))
1150    DEF FNDP3=FNEP3*(P2(2)+FNGT7(P2(3)))
1160    DEF FNDP4=FNEP3*(FNGT7(TIME(K))-FNGT7(P2(3)))
1170    DEF FNDP5=FNEP3*(FNGT9(TIME(K))-FNGT9(P2(3)))
1180    DEF   FNGT1(TM)=FNP51+FNP52*FNCO1*(P2(2)*FNCO2(TM)-2*PAI*FNSI2(TM))
1190    DEF   FNGT2(TM)=FNP52*2*PAI/FNCK1/FNCK1*(4*PAI*P2(2)*FNCO2(TM)+FNCK2*FNS
I2(TM))
1200    DEF   FNGT3(TM)=FNP52*FNCO1*(P2(2)*2*PAI*FNSI2(TM)+4*PAI*PAI*FNCO2(TM))
1210    DEF   FNGT5(TM)=1/2-1/2*FNCO1*(P2(2)*FNCO2(TM)-2*PAI*FNSI2(TM))
1220    DEF     FNEP1=EXP(-P2(2)*(TIME(K)-P2(3)))
1230    DEF     FNEP2=FNGT1(TIME(K))/FNGT1(P2(3))*FNEP1
1240    DEF     FNEP3=-P2(1)*FNEP2
1250    DEF       FNP51=(1+P2(5))/2
1260    DEF       FNP52=(1-P2(5))/2
1270    DEF       FNTM1(TM)=2*PAI*(TM-P2(4))
1280    DEF       FNCO1=P2(2)/FNCK1
1290    DEF       FNCO2(TM)=COS(FNTM1(TM))
1300    DEF       FNSI2(TM)=SIN(FNTM1(TM))
1310    DEF         FNGT6(TM)=FNGT2(TM)/FNGT1(TM)
1320    DEF         FNGT7(TM)=FNGT3(TM)/FNGT1(TM)
1330    DEF         FNGT9(TM)=FNGT5(TM)/FNGT1(TM)
1340    DEF           FNCK1=P2(2)*P2(2)+4*PAI*PAI
1350    DEF           FNCK2=P2(2)*P2(2)-4*PAI*PAI
```

Program 3. The different parts of the program for von Bertalanffy model (type-2) from Program 2.

```
1120    DEF FNDP1=1/(1+FNEP1)

1140    DEF FNDP2=P2(1)*FNEP1*(TIME(K)-P2(3))*FNDP1*FNDP1
1150    DEF FNDP3=P2(1)*FNEP1*(-P2(2))*FNDP1*FNDP1

10010  DATA 25
10020  DATA -4,1,2,-3.5,3,2,-3,4,3,-2.8,7,3,-2.4,7,3
10030  DATA -2,13,4,-1.7,17,3,-1.4,18,3,-1,25,4,-.7,38,4
10040  DATA -.3,39,5,0,48,4,.3,61,4,.6,62,4,.8,66,5
10050  DATA 1,75,4,1.3,81,3,1.7,82,3,2,87,3,2.3,92,4
10060  DATA 2.8,93,3,3,96,3,3.5,97,2,4,98,3,5,99,3
10070  DATA 100,1,0
```

Program 4-a. The different parts of the program for logistic model (type-0) from Program 1. (DATA : The example of the artificial data in Table 8).

```
1120    DEF FNDP1=1/(1+FNEP1)

1140    DEF FNDP2=FNEP2*(FNFT1(TIME(K))-FNFT1(P2(3)))*FNDP1*FNDP1
1150    DEF FNDP3=-FNEP3*FNFT3(P2(3))*FNDP1*FNDP1
1160    DEF FNDP4=FNEP3*(FNFT4(TIME(K))-FNFT4(P2(3)))*FNDP1*FNDP1
1170    DEF FNDP5=FNEP3*(FNFT5(TIME(K))-FNFT5(P2(3)))*FNDP1*FNDP1

10070  DATA 100,1,0,0,0
```

Program 4-b. The different parts of the program for logistic model (type-1) from Program 2. (DATA is omitted : same as Program 4-a).

```
1120     DEF FNDP1=EXP(-FNEP1)

1140     DEF FNDP2=P2(1)*FNEP1*(TIME(K)-P2(3))*FNDP1
1150     DEF FNDP3=P2(1)*FNEP1*(-P2(2))*FNDP1

10010   DATA 20
10020   DATA -1.3,1,3,-1,8,3,-.8,16,2,-.5,19,3,-.3,24,3
10030   DATA -.1,29,4,0,40,4,.2,50,2,.4,55,3,.7,56,5
10040   DATA 1,67,3,1.2,77,3,1.5,81,2,1.8,82,3,2,88,2
10050   DATA 2.3,92,3,2.7,92,3,3,94,3,4,98,2,5,98,3
10060   DATA 100,1,0
```

Program 5-a. The different parts of the program fon GOMPERTZ model (type-0) from Program 1. (DATA: The example of the artificial data in Table 9).

```
1120     DEF FNDP1=EXP(-FNEP1)

1140     DEF FNDP2=FNEP2*(FNFT1(TIME(K))-FNFT1(P2(3)))*FNDP1
1150     DEF FNDP3=-FNEP3*FNFT3(P2(3))*FNDP1
1160     DEF FNDP4=FNEP3*(FNFT4(TIME(K))-FNFT4(P2(3)))*FNDP1
1170     DEF FNDP5=FNEP3*(FNFT5(TIME(K))-FNFT5(P2(3)))*FNDP1

10060   DATA 100,1,0,0,0
```

Program 5-b. The different parts of the program for GOMPERTZ model (type-1) from Program 2. (DATA is omitted: same as Program 5-a).

Correspondence of variables

NIT	:	Number of iterations
CLAMBDA	:	λ
CNU	:	ν
N	:	Number of data
TIME(I)	:	t_i
BLENGTH(I)	:	l_{0i}
SIGMA(I)	:	σ_i
NP	:	Number of parameters
P(I)	:	θ_{old}
P2(I)	:	θ_{new}
PDELTA(I)	:	$\Delta\theta$
DIFFER(I)	:	$\frac{\partial l}{\partial \theta}$
SCALE(I)	:	S_1
HESSIAN(I, J)	:	H
GVECTOR(I)	:	g
Y1	:	Y_{old}
Y2	:	Y_{new}
YDELTA	:	ΔY

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周期関数による成長曲線の拡張と MARQUARDT 法による BASIC プログラム

赤嶺　達郎

周期関数：$f(t+1)=f(t)$ を用いて VON BERTALANFFY, logistic, GOMPERTZ 曲線の拡張を行なった. $h_1=-K\{F(t)-F(t_0)\}$, $F'=f$, $f=(1+a)/2+(1-a)/2\cdot\cos 2\pi(t-t_0)$: $a\leqq f\leqq 1$ のときそれぞれ $l=l_\infty(1-\exp h_1)$, $l=l_\infty/(1+\exp h_1)$, $l=l_\infty\exp(-\exp h_1)$ を得た.

各モデルの BASIC プログラムを赤嶺 (1985) に従がい MARQUARDT 法にて作成した. また別タイプへの拡張, パラメータの誤差解析, 本来のモデルや WALFORD の定差図法との比較, 成長率が極値をとるときについても考察した. この拡張は有効であり, プログラムは他の曲線へ容易に応用できる.