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Evaluation of Error Caused by Histogram on Estimation of Parameters for a Mixture of Normal Distributions

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Abstract

When we use histograms instead of raw data to estimate parameters by the maximum likelihood method, data has an error distributed according to a regular distribution among the width of the histogram. This influence on the estimation of parameters is evaluated by the linearized error propagation rule. Covariance is in proportion to the width squared and in inverse proportion to the number of data. Even if the number of data is large, the precision is low for small normal distributions. In practice, an adequate width will be given by the shapes of the histograms.

Key words error propagation rule, parameter estimation, mixture, normal distribution, histogram

I. Model

The mixture of normal distributions are defined as follows:

$$g=\sum_{i=1}^{n} P_iN_i, \quad \sum_{i=1}^{n} P_i=1, \tag{1.1}$$

$$N_i=\frac{1}{\sqrt{2\pi}\,\sigma_i}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_i}{\sigma_i}\right)^2\right\}. \quad n: \text{number of } N_i$$

The objective function Y is the maximum likelihood method as follows :

$$Y=-\sum_{k=1}^{T}\ln g. \quad T: \text{number of data} \tag{1.2}$$

Determine the parameter values to give the minimum Y. When T is large, we usually use histogram F_a, F_{a+h}, F_{a+2h}, ………, F_b. Then

$$Y^*=-\sum_{x=a}^{b} F_x \ln g \tag{1.3}$$

is used instead of (1.2). For (1.3) HASSELBLAD (1966) solved by the iteration

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method developed by himself and AKAMINE (1985, 1987) solved by MARQUARDT's method. It is easy to convert these programs for (1. 2) because (1. 3) includes (1. 2) formally. An example of the conversion from AKAMINE (1987)'s program is shown in Appendix A.

II. Error propagation rule

Let $y=\ln g$ in (1. 2). Then the following equation holds for the solution.

$$\frac{\partial Y}{\partial \theta}=-\sum_{k=1}^{T}\frac{\partial y}{\partial \theta}=0 \tag{2. 1}$$

When error $\delta \boldsymbol{x}$ is added to data $\boldsymbol{x}$, parameters $\boldsymbol{\theta}$ increase $\delta\boldsymbol{\theta}$. Then (2. 1) leads to

$$\delta\frac{\partial Y}{\partial \theta_i}\doteq\sum_{j=1}^{p}\frac{\partial^2 Y}{\partial \theta_i \partial \theta_j}\delta\theta_j+\sum_{k=1}^{T}\frac{\partial^2 Y}{\partial \theta_i \partial x_k}\delta x_k=0. \tag{2. 2}$$

p : number of parameters

This is as follows for matrix.

$$\boldsymbol{H}\delta\boldsymbol{\theta}+\boldsymbol{J}\delta\boldsymbol{x}=0. \tag{2. 3}$$

$$\boldsymbol{H}=\left(\frac{\partial^2 Y}{\partial \theta_i \partial \theta_j}\right)=\left(-\sum_{k=1}^{T}\frac{\partial^2 y}{\partial \theta_i \partial \theta_j}\right) \text{ : Hessian}$$

$$\boldsymbol{J}=\left(\frac{\partial^2 Y}{\partial \theta_i \partial x_k}\right)=\left(-\frac{\partial^2 y}{\partial \theta_i \partial x_k}\right) \text{ : Jacobian}$$

Where

$$\frac{\partial^2 y}{\partial \theta_i \partial \theta_j}=\frac{1}{g^2}\left(\frac{\partial^2 g}{\partial \theta_i \partial \theta_j}g-\frac{\partial g}{\partial \theta_i}\frac{\partial g}{\partial \theta_j}\right) \tag{2. 4}$$

$$\frac{\partial^2 y}{\partial \theta_i \partial x}=\frac{1}{g^2}\left(\frac{\partial^2 g}{\partial \theta_i \partial x}g-\frac{\partial g}{\partial \theta_i}\frac{\partial g}{\partial x}\right) \tag{2. 5}$$

According to (1. 1), in this paper P_n is excluded as follows :

$$P_n=1-\sum_{i=1}^{n-1}P_i. \tag{2. 6}$$

Then the concrete expression for (2. 4) and (2. 5) are as follows :

$$\frac{\partial g}{\partial x}=\sum_{i=1}^{n}P_i\,N_i\left(-\frac{x-\mu_i}{\sigma_i^2}\right) \tag{2. 7}$$

$$\frac{\partial^2 g}{\partial P_i \partial x}=N_i\left(-\frac{x-\mu_i}{\sigma_i^2}\right)-N_n\left(-\frac{x-\mu_n}{\sigma_n^2}\right) \tag{2. 8}$$

$$\frac{\partial^2 g}{\partial \mu_i \partial x}=P_i\,N_i\frac{-(x-\mu_i)^2+\sigma_i^2}{\sigma_i^4} \tag{2. 9}$$

$$\frac{\partial^2 g}{\partial \sigma_i \partial x}=P_i\,N_i\frac{-(x-\mu_i)^3+3(x-\mu_i)\sigma_i^2}{\sigma_i^5} \tag{2. 10}$$

$\partial g/\partial\theta$ and $\partial^2 g/\partial\theta_i\partial\theta_j$ are shown in AKAMINE (1987).

(2.3) leads to

$$\partial\boldsymbol{\theta} = -\boldsymbol{H}^{-1}\boldsymbol{J}\partial\boldsymbol{x}. \tag{2.3'}$$

Then we get

$$<\partial\boldsymbol{\theta}^t\partial\boldsymbol{\theta}> = \boldsymbol{H}^{-1}\boldsymbol{J}<\partial\boldsymbol{x}^t\partial\boldsymbol{x}>^t\boldsymbol{J}\boldsymbol{H}^{-1} \tag{2.11}$$

$< \ >$: Expected value

${}^t\boldsymbol{A}$: Transfomed matrix of $\boldsymbol{A}$

This is the linearized error propagation rule. ∂x distributes according to a regular distribution among h (width of histogram) independently of x. Therefore,

$$<\partial x_i\partial x_j> = \begin{cases} \dfrac{h^2}{12} & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{2.12}$$

Then, let $\boldsymbol{B} = \boldsymbol{J}^t\boldsymbol{J}$. It leads to

$$<\partial\boldsymbol{\theta}^t\partial\boldsymbol{\theta}> = \frac{h^2}{12}\boldsymbol{H}^{-1}\boldsymbol{B}\boldsymbol{H}^{-1} = \frac{h^2}{12T}\hat{\boldsymbol{H}}^{-1}\hat{\boldsymbol{B}}\hat{\boldsymbol{H}}^{-1} \tag{2.13}$$

Where

$$\boldsymbol{H} = T\hat{\boldsymbol{H}} = T(\hat{h}_{ij}), \quad \boldsymbol{H}^{-1} = \hat{\boldsymbol{H}}^{-1}/T = (\hat{h}^{-1}{}_{ij})/T,$$

$$\boldsymbol{B} = T\hat{\boldsymbol{B}} = T(\hat{b}_{ij}).$$

And, substituting (2.3) for

$$\delta Y \doteqdot \frac{1}{2}{}^t\partial\boldsymbol{\theta}\boldsymbol{H}\partial\boldsymbol{\theta}, \tag{2.14}$$

and cleaning, we get

$$<\delta Y> \doteqdot \frac{h^2}{24}\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\hat{h}^{-1}{}_{ij}\hat{b}_{ij}\right). \tag{2.15}$$

Although $<\partial\boldsymbol{\theta}^t\partial\boldsymbol{\theta}>$ decreases when T increases, $<\delta Y>$ is not concerned with T because $\boldsymbol{B}$ is in proportion to T and $\boldsymbol{H}^{-1}$ is in inverse proportion to T (Fig. 1).

When $h=0$, covariance matrix $\boldsymbol{V}$ is as follows according to AKAMINE (1985).

$$\boldsymbol{V} = <\Delta\boldsymbol{\theta}^t\Delta\boldsymbol{\theta}> \sim \boldsymbol{H}^{-1} \tag{2.16}$$

When $h \neq 0$, $\partial\theta$ is independent of $\Delta\theta$. Therefore,

$$\boldsymbol{V} = <\Delta\boldsymbol{\theta}^t\Delta\boldsymbol{\theta}> + <\partial\boldsymbol{\theta}^t\partial\boldsymbol{\theta}>. \tag{2.17}$$

The following approximations hold.

$$\boldsymbol{H} = \left(-\sum_{k=1}^{T}\frac{\partial^2 y}{\partial\theta_i\partial\theta_j}\right) \tag{2.3}$$

$$\doteqdot \left(-\sum_{x=a}^{b}F_x\frac{\partial^2 y}{\partial\theta_i\partial\theta_j}\right) \tag{2.18}$$

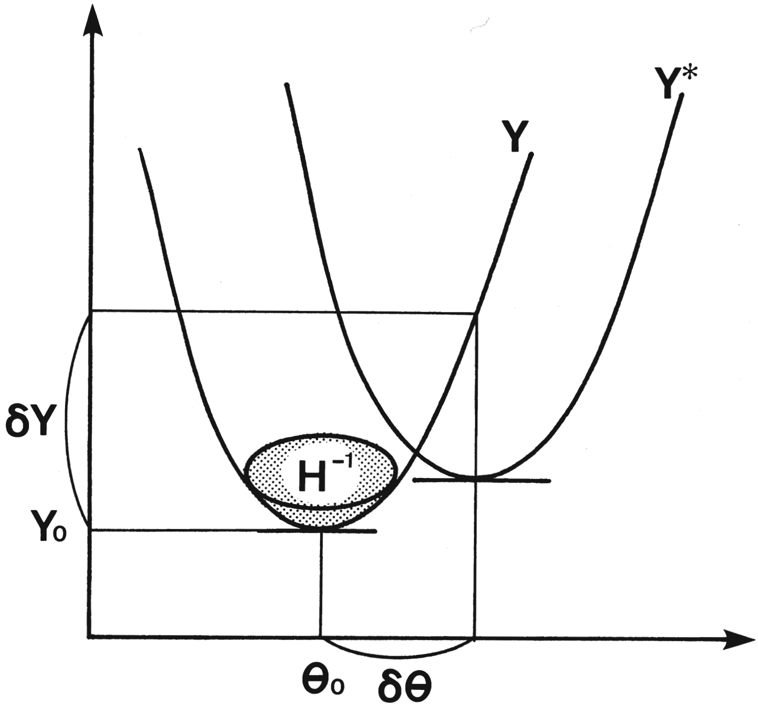

Fig. 1. An illustration of the neighborhood of the solution.
Y : objective function θ : parameter

$$\doteqdot -T\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial\theta_i\partial\theta_j}g\,dx \tag{2.19}$$

$$\boldsymbol{B}=\left(\sum_{k=1}^{T}\frac{\partial^2 y}{\partial\theta_i\partial x_k}\cdot\frac{\partial^2 y}{\partial\theta_j\partial x_k}\right) \tag{2.20}$$

$$\doteqdot\left(\sum_{x=a}^{b}F_x\frac{\partial^2 y}{\partial\theta_i\partial x}\cdot\frac{\partial^2 y}{\partial\theta_j\partial x}\right) \tag{2.21}$$

$$\doteqdot T\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial\theta_i\partial x}\cdot\frac{\partial^2 y}{\partial\theta_j\partial x}g\,dx \tag{2.22}$$

(2. 18) and (2. 21) have error caused by the histogram. Precision of (2. 19) and (2. 22) is dependent on the fitness of data for g. In practice, the trapezoidal rule is enough for the calculation of (2. 19) and (2. 22). For an improper integral of infinite interval as a normal distribution, the trapezoidal rule is the best (e. g. MORI 1974).

III. Experiment

Data made from normal random numbers (YAMAUTI 1972) for this experiment are shown in Table 1. This is raw data ($h=0$). N_4 and N_5 have bias because the number of data is small. The data for the width of histogram $h=1, 2, 3, 4$ are shown in Table 2. The computer programs used for this experiment are Appendix A of

Table 1-a. Raw data of N_1 (10, 3.5).

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Total
1				1			1				2
2					2						2
3	1	1					1		2		5
4		2		1		1	2		2	1	9
5	2			1		5	2	1		1	12
6	3	1	3	1	1			2		3	14
7	1	1	1	1	3	3	2	1	1	4	18
8	1	2		3		2	1	2	2	2	15
9	1	4	1	1	4	5		1	3		20
10	4	3	1		2	1	1	4	4	1	21
11	1	7	1	2	3	4			3	4	25
12	2	1	1	5		1	1	1	3	4	19
13	1	1	3	2	3		1			1	12
14	1	2		1	1			2	1	1	9
15	2				1	1	2				6
16		1				1		1	1	2	6
17				1	1						2
18							1				1
19		1									1
20											
21											
22	1										1
Total	21	27	11	20	21	24	15	15	22	24	200

Table 1-b. Raw data of N_2 (22, 3.5).

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Total
13									1		1
14											
15											
16	1			2				1			4
17		1	1	1	2	2	1				8
18	1	1		5			2	1	1	3	14
19	3	1	1	2		3	1		1	1	13
20			1	4	1	1		2	3	1	13
21	1	3	3	4	4	2	1	2	1		21
22	1		2	1	3	2	2	3	2		16
23	2	3	2	3	3		1			1	15
24	3	3	2	2	2	1	1	1		1	16
25		3	1	2	2	1	1	2			12
26	1		1	2						1	5
27	1			1			1		2	1	6
28	2				1			1		1	5
29							1				1
Total	16	15	14	29	18	12	12	13	11	10	150

Table 1-c. Raw data of N_3 (33, 3.0).

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Total
25				1							1
26											
27							1				1
28					1				1	1	3
29							1	1			2
30		2	1	1		1	1	1	1		8
31		1		1				3	3	1	9
32	1	1	1	1	1			1	1	1	8
33	1			2		1		1		2	7
34	2		1	2	2	2			2		11
35		1		1			1	1	1		5
36	4				1		2	1	1		9
37	1				1				2		4
38		1				1	1				3
39			1			1				1	3
40				1							1
Total	9	6	4	10	6	6	7	9	12	6	75

Table 1-d. Raw data of N_4 (43, 2.5).

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Total
36										1	1
37					1	1					2
38					1	1		1			3
39				1	1	1	1		2		6
40					1		2	1	2		6
41	1			1	1			1			4
42						4	3	2		1	10
43	1		1	1	2			1	1		7
44				1			1		1	1	4
45		2				1	1				4
46							1		1		2
47	1										1
Total	3	2	1	4	7	8	9	6	7	3	50

Table 1-e. Raw data of N_5 (51, 2.5).

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Total
45				1							1
46								1			1
47							1				1
48	1				2	1					4
49					1			1			2
50							1				1
51		1	2			1				2	6
52						1	1	1			3
53	1		1	1							3
54		1				1					2
55						1					1
Total	2	2	3	2	3	5	3	3		2	25

Table 2-a. Data for histogram h=1.

x	F_x	x	F_x	x	F_x	x	F_x
1.5	2	15.5	6	29.5	3	43.5	7
2.5	2	16.5	10	30.5	8	44.5	4
3.5	5	17.5	10	31.5	9	45.5	5
4.5	9	18.5	15	32.5	8	46.5	3
5.5	12	19.5	14	33.5	7	47.5	2
6.5	14	20.5	13	34.5	11	48.5	4
7.5	18	21.5	21	35.5	5	49.5	2
8.5	15	22.5	17	36.5	10	50.5	1
9.5	20	23.5	15	37.5	6	51.5	6
10.5	21	24.5	16	38.5	6	52.5	3
11.5	25	25.5	13	39.5	9	53.5	3
12.5	19	26.5	5	40.5	7	54.5	2
13.5	13	27.5	7	41.5	4	55.5	1
14.5	9	28.5	8	42.5	10		

Table 2-b. Data for histogram h=2.

x	F_x	x	F_x	x	F_x	x	F_x
1	2	15	15	29	11	43	17
3	7	17	20	31	17	45	9
5	21	19	29	33	15	47	5
7	32	21	34	35	16	49	6
9	35	23	32	37	16	51	7
11	46	25	29	39	15	53	6
13	32	27	12	41	11	55	3

Table 2-c. Data for histogram $h=3$.

x	F_x	x	F_x	x	F_x	x	F_x
1.5	4	16.5	26	31.5	25	46.5	10
4.5	26	19.5	42	34.5	23	49.5	7
7.5	47	22.5	53	37.5	22	52.5	12
10.5	66	25.5	34	40.5	20	55.5	3
13.5	41	28.5	18	43.5	21		

Table 2-d. Data for histogram $h=4$.

x	F_x	x	F_x	x	F_x	x	F_x
2	9	18	49	34	31	50	13
6	53	22	66	38	31	54	9
10	81	26	41	42	28		
14	47	30	28	45	14		

AKAMINE (1987) and Appendix A of this paper (MARQUARDT's method). The initial values for these calculations are the original values of data. The data of $h=5$ did not converge by using either this program or HASSELBLAD (1966)'s iteration method (Appendix B of AKAMINE 1987) because of error (when $g<0$ caused by $P_i<0$, LOG (g) cannot be defined). Results are shown in Table 3 and Fig. 2. The solution of $h=0$ is different from the original values because of the bias of N_4 and N_5.

Table 3. Results of the calculations.

		Original values	Solution				
			data-0	data-1	data-2	data-3	data-4
σ	1	3.5	3.46922	3.45241	3.51287	3.62300	3.42647
	2	3.5	3.27365	3.45002	3.15141	3.30452	4.16409
	3	3.0	3.06854	2.94871	6.57654	3.68612	4.45196
	4	2.5	4.44999	4.68817	1.31251	3.59731	3.68281
	5	2.5	1.44114	1.53241	2.28118	1.88585	2.15709
μ	1	10	9.72112	9.73426	9.73179	9.87249	9.30553
	2	22	21.7277	21.8949	21.4799	21.9786	21.3899
	3	33	32.5708	32.6374	35.3554	33.0844	33.7139
	4	43	41.5331	41.1827	43.1750	42.2803	41.9722
	5	51	52.4835	52.5322	51.3989	52.2755	51.6600
P	1	.40	.385945	.383253	.385350	.392498	.356020
	2	.30	.306997	.315608	.276731	.294951	.338307
	3	.15	.120122	.105166	.275346	.151181	.158263
	4	.10	.157461	.166713	.0232848	.123567	.106954
Y_0		1918.08	1911.52	1911.75	1913.16	1912.57	1915.17
∂Y		(6.56)	0	0.23	1.64	1.05	3.65
$<\partial Y>$			0	0.20	0.80	1.80	3.20

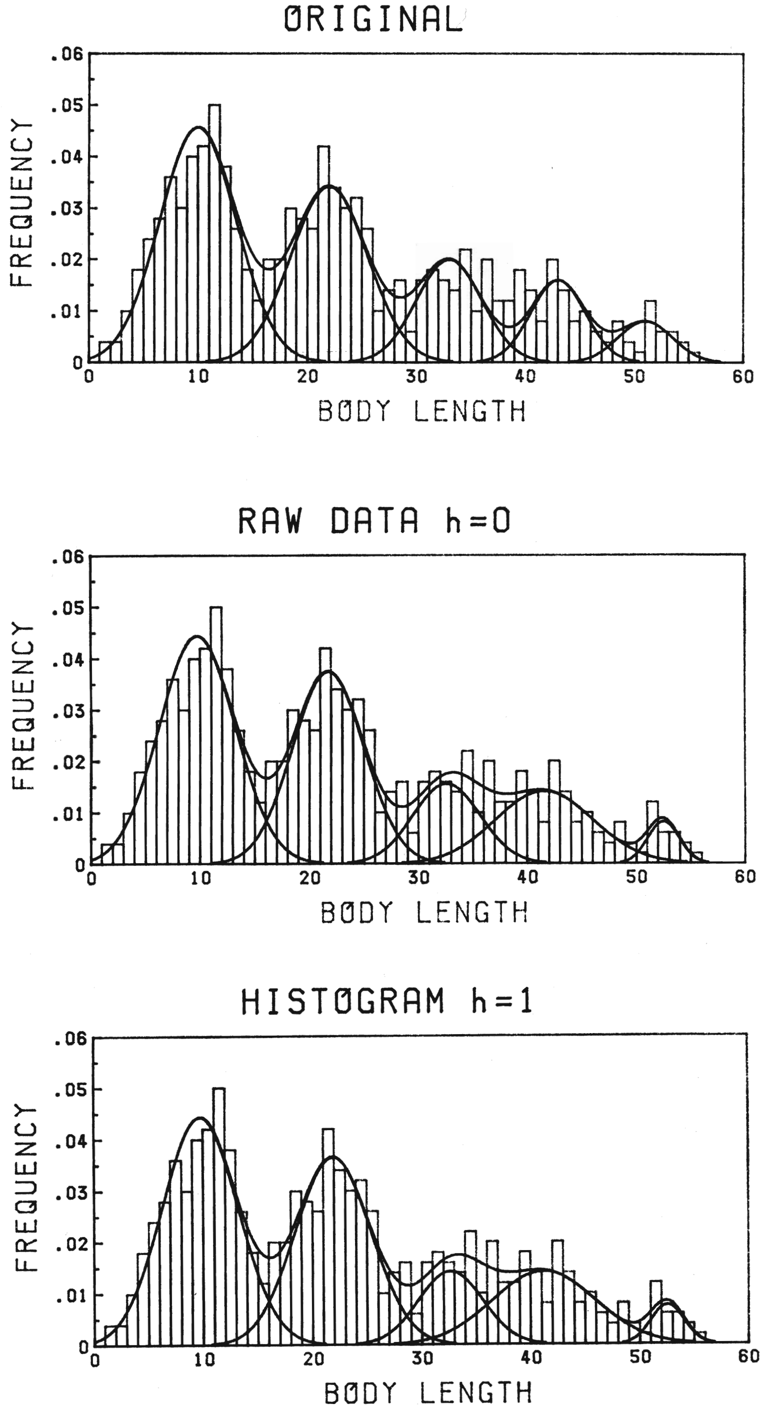

Fig. 2–a. Results of the calculations (see Table 3).

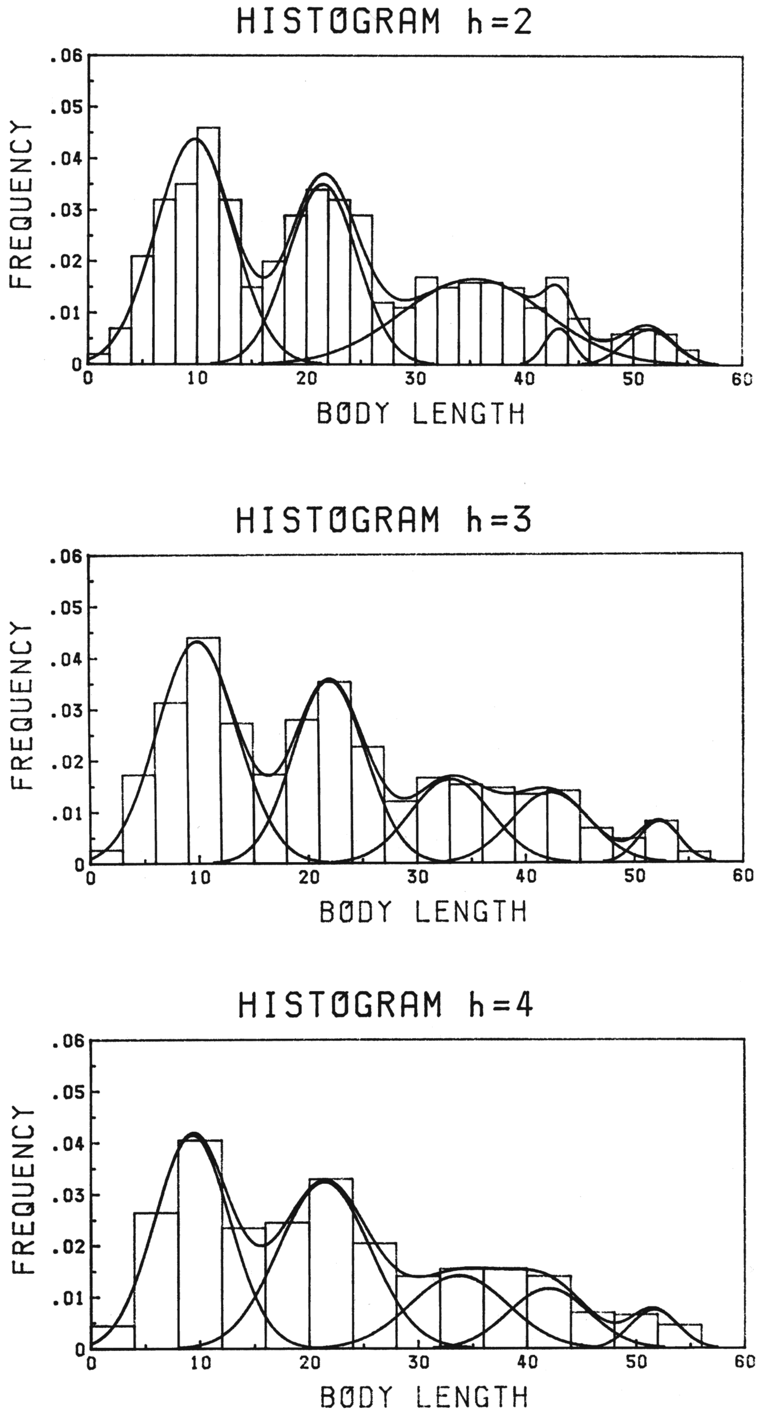

Fig. 2-b. Continued.

Table 4. $\boldsymbol{H}^{-1}$ and $\boldsymbol{R}(\times 100)$ of the solution for data-0.

		σ					μ					P			
		1	2	3	4	5	1	2	3	4	5	1	2	3	4
σ	1	85.1	-79.9	93.8	-37.9	7.8	21.9	4.8	-.7	6.7	-.2	33.6	-63.2	123.6	-56.2
	2	-48	324.5	-561.0	239.9	-49.7	-38.1	8.0	-.2	-42.2	1.3	-65.3	196.3	-732.5	352.9
	3	19	-58	2899.3	-1894.6	430.2	47.1	-53.6	161.2	312.5	-11.4	83.4	-439.7	4067.7	-2592.5
	4	-8	27	-72	2421.7	-740.8	-19.2	25.6	-200.5	-316.3	21.1	-34.1	199.3	-3638.7	2698.5
	5	3	-9	25	-46	1049.4	3.9	-5.4	50.8	70.4	-21.1	7.0	-41.7	918.4	-758.5
μ	1	60	-53	22	-10	3	15.8	1.8	-.3	3.4	-.1	14.4	-28.5	62.1	-28.4
	2	24	21	-47	24	-8	21	4.6	-.9	-4.4	.1	2.6	5.6	-68.8	37.1
	3	-2	-0	60	-81	31	-2	-9	25.3	30.2	-1.4	-.5	-3.9	316.2	-248.7
	4	10	-33	82	-91	31	12	-29	85	50.2	-1.8	6.0	-34.8	554.3	-393.5
	5	-2	6	-19	39	-60	-2	6	-25	-24	1.2	-.2	1.1	-25.0	21.3
P	1	49	-49	21	-9	3	49	16	-1	11	-2	54.9	-67.2	89.7	-70.5
	2	-50	80	-60	30	-9	-53	19	-6	-36	7	-67	186.0	-591.1	270.9
	3	16	-49	91	-89	34	19	-39	76	94	-28	15	-52	6879.4	-4607.7
	4	-10	34	-83	94	-40	-12	30	-85	-95	33	-16	34	-95	3399.6

Table 5. ***B*** of the solution for data-0.

		σ					μ					P			
		1	2	3	4	5	1	2	3	4	5	1	2	3	4
σ	1	4.15065	.99425	−.00250	−.00009	−.00000	−.27526	−.44838	.00108	.00003	.00000	−.02529	.03625	−.01113	−.00019
	2		2.76157	.56377	−.01146	−.00000	.52881	.17963	−.34101	.01908	.00000	.94910	−1.29988	.73467	−.35242
	3			.86686	−.06476	−.00010	−.00090	.33773	−.03358	−.10308	.00005	−.00159	.93064	−5.18983	2.15586
	4				.44043	.90692	−.00003	−.02460	.07684	.14502	−.13713	4.52386	4.30309	7.02376	3.8946
	5					9.01619	−.00000	−.00000	−.00002	.40319	.75027	21.2253	21.2253	21.2252	25.1985
μ	1						1.39148	−.08234	.00035	.00001	.00000	1.11289	−1.39775	−.00334	−.00006
	2							1.32863	−.06039	.01676	.00000	.78998	.15947	−2.63435	−.23755
	3								.46985	−.08413	.00001	.00122	.65591	−2.06995	.29882
	4									.20938	−.05236	5.66988	5.7857	5.63624	6.53092
	5										2.73184	−.18643	−.18643	−.18638	−.22137
P	1											585.83	562.59	575.52	683.26
	2												604.12	545.91	681.84
	3													702.74	643.99
	4														843.90

Table 6. $\langle \partial\boldsymbol{\theta}'\partial\boldsymbol{\theta} \rangle$ and $\boldsymbol{R}(\times 100)$ calculated by $\boldsymbol{H}^{-1}$ and $\boldsymbol{B}$.

		σ					μ					P			
		1	2	3	4	5	1	2	3	4	5	1	2	3	4
σ	1	32.0	-43.7	63.1	-25.8	7.6	10.3	1.2	-.5	4.4	-.2	15.9	-33.8	82.1	-37.8
	2	-56	188.4	-388.9	170.2	-50.2	-20.8	8.0	-.4	-28.7	1.5	-36.3	123.3	-504.0	246.9
	3	26	-67	1813.3	-1221.2	425.7	32.0	-35.2	94.4	190.4	-13.0	57.8	-297.0	2576.8	-1639.1
	4	-12	33	-76	1407.6	-750.7	-13.2	17.5	-117.2	-180.8	23.4	-24.0	138.5	-2206.6	1635.6
	5	3	-9	26	-52	1506.7	3.9	-5.4	49.7	64.7	-34.9	7.0	-41.7	906.6	-776.0
μ	1	83	-69	34	-16	5	4.8	.3	-.2	2.2	-.1	6.7	-15.3	41.6	-19.4
	2	17	45	-64	36	-11	10	1.6	-.6	-2.9	.2	.2	5.5	-45.5	24.9
	3	-3	-1	61	-86	35	-3	-14	13.2	16.4	-1.5	-.3	-2.7	179.1	-142.5
	4	15	-41	87	-94	32	20	-44	88	26.4	-2.0	4.1	-23.2	319.2	-225.5
	5	-4	10	-27	56	-80	-5	11	-38	-35	1.2	-.2	1.3	-28.1	24.1
P	1	85	-80	41	-19	5	93	5	-3	24	-6	10.8	-25.9	75.1	-35.1
	2	-64	96	-75	39	-12	-75	46	-8	-48	12	-84	87.4	-385.8	198.9
	3	23	-57	95	-92	36	30	-56	77	97	-39	36	-64	4100.1	2780.1
	4	-15	40	-86	98	-45	-20	43	-88	-98	48	-24	48	-97	1995.1

$\boldsymbol{H}^{-1}$ and $\boldsymbol{B}$ of the solution for $h=0$ calculated by (2.3) and (2.20) are shown in Tables 4 and 5. $<\partial\boldsymbol{\theta}^t\partial\boldsymbol{\theta}>$ calculated by (2.13) is shown in Table 6. Scaling is not done for these value. On the other hand,

$$<\partial Y>=\frac{4.78399}{24}h^2 \doteqdot 0.20\,h^2.$$

Seeing ∂Y of Table 3, for $h=1$ they are almost equal, but for $h=2$ they are far apart. For $h=2$ the linearized approximation (2.3) does not hold. It appears in Fig. 2.

In order to calculate (2.19) and (2.22) to 3 significant figures by the trapezoidal rule, it is enough for this data to let $x=-10\sim70$, $h=0.5$.

IV. Consideration

When T (number of data) increases, $<\Delta\boldsymbol{\theta}^t\Delta\boldsymbol{\theta}>$ and $<\partial\boldsymbol{\theta}^t\partial\boldsymbol{\theta}>$ decreases. $<\partial\boldsymbol{\theta}^t\partial\boldsymbol{\theta}>$ can be omitted if h (width of histogram) is small to some degree. The linearized approximation is not efficient because the non-linearity is too strong. In practice, the shapes of histograms are enough to judge the adequate h. Although it is the best to estimate for raw data ($h=0$), it is time consuming. It is better to decide h according to T, the precision of data and the purpose.

Although at the data in this paper $\sigma_{\text{young}}>\sigma_{\text{old}}$, at body length of fishes $\sigma_{\text{young}}<\sigma_{\text{old}}$ in general. For the latter the precision of estimated values about N_{old} may be lower. Be careful even if T is large because the precision for N_{old} is low.

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多重正規分布のパラメータ推定における ヒストグラムによる誤差の影響

赤 嶺 達 郎

最尤法でパラメータ推定を行う場合，生データの代わりにヒストグラムを用いると，データに誤差が加わる．これはヒストグラムの幅で一様分布していると考えられる．線型近似した誤差伝播則よりパラメータの推定値への影響を評価できる．共分散は幅の二乗に比例し，データ数に反比例する．データ数が多くても比率の小さい正規分布では精度が悪い．実際上はヒストグラムの形より適正な幅を判定できる．

Appendix A. The BASIC program to estimate parameters of a mixture of normal distributions for raw data. Changing parts from Appendix A of Akamine (1987).

```
10      '---------------------------------------------------------
20      '   Polymodal model by Marquardt's method for raw data
30      '
40      '                   from Appendix A of Akamine(1987)
50      '                                             1985-10-23
60      '---------------------------------------------------------
2060    '
2070    PRINT "Number of raw data              =";NOX
2080    '
2140    DIM X(NOX),GX(NOX),DIFFER(NP),BND(NND,NOX)
2180    '
2190    FOR K=1 TO NOX
2200       '
2210       READ X(K)
2220       PRINT "X(";K;")=";X(K),
2230       '
2260    PRINT "Sum of data =";NOX
4100    FOR K=1 TO NOX
4110       G1=GX(K) : F7=1/G1 : F6=F7/G1
6050    FOR K=1 TO NOX
6110       Y2=Y2-LOG(F1)
```