日水研報告，(38)：187−200，1988.
Bull. Jap. Sea Reg. Fish. Res. Lab., (38)：187−200，1988.

Estimation of Parameters for RICHARDS Model

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Abstract

AKAMINE (1986)'s BASIC program by MARQUARDT's method was rewritten for RICHARDS model and its expanded model by the periodic function. For 0.9∼1.1 the "LOG" function is corrected by TAYLOR series. Data estimated to be negative are cut off. AIC judges the effect of adding n to the parameters. RICHARDS model is not so important in practice but it is important theoretically.

Key words RICHARDS, MARQUARDT, TAYLOR series, AIC, BASIC program

I. Introduction

AKAMINE (1986) estimated parameters by MARQUARDT's method for VON BERTALANFFY, logistic and GOMPERTZ models and their expanded models by the periodic function. RICHARDS model includes these three models. In this paper, estimation of parameters for RICHARDS model and correction of the "LOG" function in the calculation will be described.

II. RICHARDS model

1. Model

RICHARDS model is defined in the differential equation as follows :

$$\frac{dl}{dt}=Kl\frac{1-\left(\frac{l}{l_\infty}\right)^n}{n}. \tag{2.1}$$

Then let

$$v=\frac{\left(\frac{l_\infty}{l}\right)^n-1}{n}. \tag{2.2}$$

Therefore,

$$\frac{dv}{dt}=-\frac{l_\infty{}^n}{l^{n+1}}\frac{dl}{dt}. \tag{2.3}$$

Received : October 13, 1987. Contribution A *No.* 447 from the Japan Sea Regional Fisheries Research Laboratory.

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Substitution of (2.1) and (2.2) into (2.3) gives

$$\frac{dv}{dt}=-Kv. \tag{2.4}$$

A general solution of this differential equation is

$$v=e^h,\quad h=-Kt+c,\quad c: \text{integral constant.} \tag{2.5}$$

Substitution of (2.5) into (2.2) gives

$$l=\frac{l_\infty}{(1+ne^h)^{\frac{1}{n}}},\quad h=-Kt+c. \tag{2.6}$$

This is the general solution of RICHARDS model.

The initial condition :

$$\text{when } t=t_0,\quad l=\frac{l_\infty}{(1+n)^{\frac{1}{n}}} \tag{2.7}$$

gives the particular solution :

$$l=\frac{l_\infty}{(1+ne^h)^{\frac{1}{n}}},\quad h=-K(t-t_0). \tag{2.8}$$

On the other hand, the initial condition :

$$\text{when } t=0,\quad l=l_0 \tag{2.9}$$

gives the particular solution :

$$l=\frac{l_\infty}{\{1+(p^n-1)e^h\}^{\frac{1}{n}}},\quad p=\frac{l_\infty}{l_0},\quad h=-Kt. \tag{2.10}$$

Note that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=\lim_{n\to 0}(1+nx)^{\frac{1}{n}}=e^x, \tag{2.11}$$

$$\lim_{n\to 0}\frac{y^n-1}{n}=\ln y. \tag{2.12}$$

Then (2.8) corresponds to three models as follows :

$$\begin{cases} n=-1 & : \text{VON BERTALANFFY model} \\ n\to 0 & : \text{GOMPERTZ model} \\ n=1 & : \text{logistic model} \end{cases}$$

Relation of t_0 and l_0 is

$$t_0=\frac{1}{K}\ln\frac{p^n-1}{n},\quad \text{when } n\to 0,\quad t_0=\frac{1}{K}\ln(\ln p). \tag{2.13}$$

Although (2.9) is more general than (2.7) as an initial condition, (2.7) is usually used in fishery population dynamics and is easier to treat in calculation. (2.13) combines (2.8) and (2.10).

2. Property

Setting $l''=0$, we find that

$$1-(n+1)\left(\frac{l}{l_\infty}\right)^n=0. \tag{2.14}$$

When $n>-1$, this equation has a solution corresponding to (2.7). Namely, when $n>-1$, t_0 is an inflection point.

Outlines of this model are as follows : First, (2.10) gives

$$\lim_{n\to\infty} l=l_0, \tag{2.15}$$

$$\lim_{n\to-\infty} l=\begin{cases} l_0 & (t=0) \\ l_\infty & (t>0) \end{cases} \tag{2.16}$$

These are shown in Fig. 1. On the other hand, (2.8) gives

$$\lim_{n\to\infty} l=l_\infty, \tag{2.17}$$

$$\lim_{n\to-\infty} l=\begin{cases} 0 & (t=t_0') \\ l_\infty & (t>t_0') \end{cases}, \quad t_0'=t_0+\frac{\ln(-n)}{K}. \tag{2.18}$$

Namely, (2.8) has intersection t_0' with the transverse axis ($l=0$) when $n<0$. These are shown in Fig. 2. When $n<0$, (2.8) is rewritten as

$$l=l_\infty(1-e^h)^m,\quad h=-K(t-t_0'),\quad m=-\frac{1}{n}. \tag{2.8'}$$

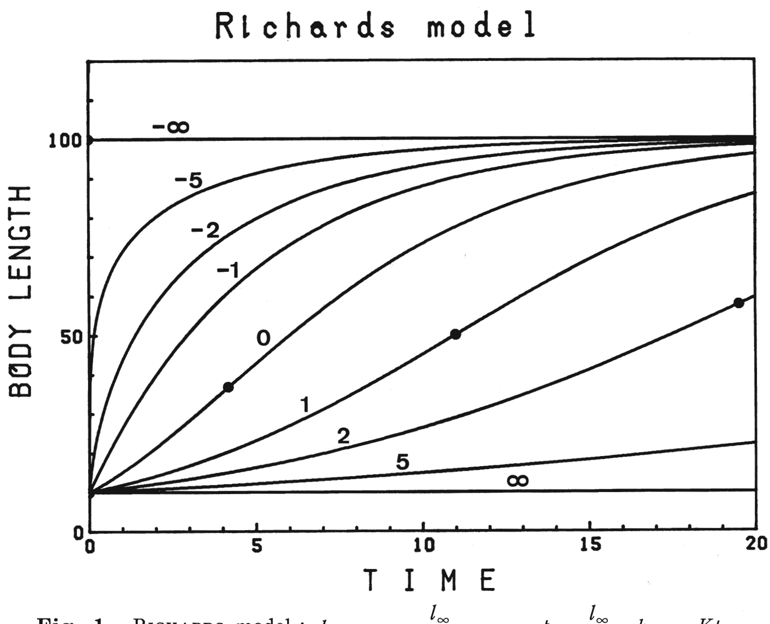

Fig. 1. RICHARDS model : $l=\dfrac{l_\infty}{\{1+(p^n-1)e^h\}^{\frac{1}{n}}}$, $p=\dfrac{l_\infty}{l_0}$, $h=-Kt$.
$l_\infty=100$, $K=0.2$, $l_0=10$, $n=-\infty$, -5, -2, -1, 0, 1, 2, 5, ∞.

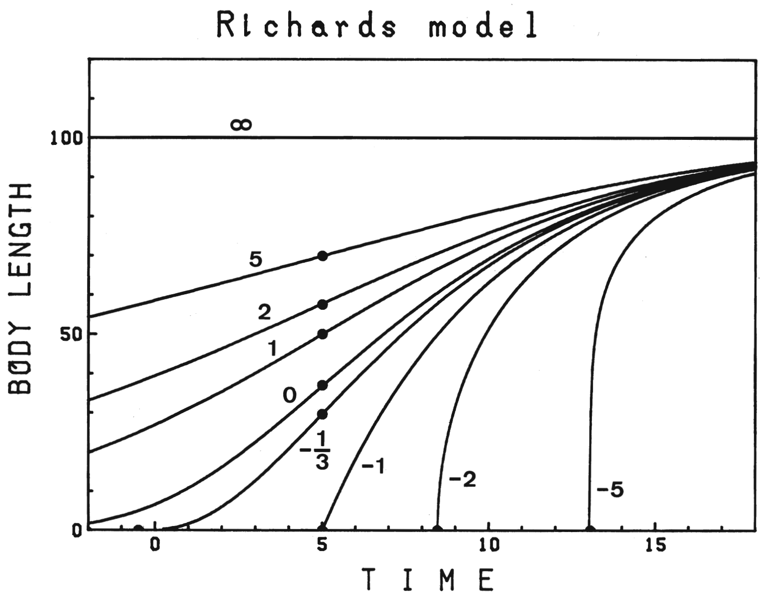

Fig. 2. RICHARDS model : $l=\dfrac{l_\infty}{(1+ne^h)^{\frac{1}{n}}}$, $h=-K(t-t_0)$.
$l_\infty=100$, $K=0.2$, $t_0=5$, n$=-5$, -2, -1, $-1/3$, 0, 1, 2, 5, ∞.

This is called the generalized VON BERTALANFFY model. Specifically, it is generally used as a body weight growth model when $n=-1/3$.

3. Expanded model by the periodic function

This expansion is difined as follows :

$$K \rightarrow Kf(t), \quad f(t+1)=f(t) \tag{2.19}$$

AKAMINE (1986) presented two models defined as follows in (2.4) :

$$\frac{dv}{dt}=-Kf(t)v \qquad \text{(type–1)} \tag{2.20}$$

$$\frac{dv}{dt}=-Kf(t)e^h \qquad \text{(type–2)} \tag{2.21}$$

In this paper, only the type–1 model is discussed because the type–1 is more natural than the type–2. The particular solution of (2.20) with the initial condition as (2.7) is

$$l=\frac{l_\infty}{(1+ne^h)^{\frac{1}{n}}}, \quad h=-K\{F(t)-F(t_0)\}, \quad F'(t)=f(t). \tag{2.22}$$

AKAMINE (1986) used the following periodic function.

$$f(t)=\frac{1+a}{2}+\frac{1-a}{2}\cos 2\pi(t-t_1) \tag{2.23}$$

$$F(t)=\frac{1+a}{2}t+\frac{1-a}{4\pi}\sin 2\pi(t-t_1) \tag{2.24}$$

The inflection points are given by the following equation.

$$h'^2(1-e^h)+h''(1+ne^h)=0 \tag{2.25}$$

This equation is the general equation of eqation (33), (53) and (63) of AKAMINE (1986) and can be solved by NEWTON's method.

III. Estimation of parameters

1. MARQUARDT's method

This is the expanded NEWTON's method. Let Y be the objective function, $\boldsymbol{\theta}$ be parameters. It is expressed as follows in the case of searching the minimal point.

$$(\boldsymbol{H}+\lambda\boldsymbol{I})\Delta\boldsymbol{\theta}=\boldsymbol{g},\ \boldsymbol{H}=\left(\frac{\partial^2 Y}{\partial\theta_i\partial\theta_j}\right),\ \boldsymbol{g}=-\frac{\partial Y}{\partial\boldsymbol{\theta}}. \tag{3.1}$$

$\boldsymbol{H}$: Hessian matrix
$\boldsymbol{I}$: unit matrix
$\boldsymbol{g}$: gradient vector

λ is the control factor of convergence. When $\lambda\to\infty$ it approaches the steepest descent method : $\Delta\boldsymbol{\theta}=(1/\lambda)\boldsymbol{g}$. On the other hand, when $\lambda\to 0$ it approaches NEWTON's method : $\boldsymbol{H}\Delta\boldsymbol{\theta}=\boldsymbol{g}$. Therefore, let λ be large at first, and make it small step by step to get the solution. In this paper, the simplest method is used. When $\Delta Y<0$ let $\lambda^{new}=\lambda^{old}/2$ to continue the calculation, when $\Delta Y\geq 0$ let $\lambda^{new}=\lambda^{old}*2$ and try again the same iterative routine. When $\Delta Y\geq 0$ after 10 times enlarging λ continuously, we determine it to be the solution to end the calculation.

Scaling of the parameters is necessary because MARQUARDT's method is like to the steepest descent method at first. Scaling is defined by a diagonal matrix $\boldsymbol{S}$. Then (3.1) becomes

$$(\boldsymbol{S}^{-1}\boldsymbol{H}\boldsymbol{S}^{-1}+\lambda\boldsymbol{I})\boldsymbol{S}\Delta\boldsymbol{\theta}=\boldsymbol{S}^{-1}\boldsymbol{g}.$$

Cleaning this equation, we have

$$(\boldsymbol{H}+\lambda\boldsymbol{S}^2)\Delta\boldsymbol{\theta}=\boldsymbol{g}. \tag{3.2}$$

Generally, we use

$$\boldsymbol{S}^2=\mathrm{diag}\boldsymbol{H}. \tag{3.3}$$

diag$\boldsymbol{A}$: diagonal matrix composed of only the diagonal elements of $\boldsymbol{A}$

Then it is sufficient to enlarge the diagonal elements of $\boldsymbol{H}$ by a factor $(1+\lambda)$. If $\boldsymbol{H}$ has a negative part of its diagonal elements, let $\lambda>1$ and enlarge that part by a factor $(\lambda-1)$ and the other part by a factor $(1+\lambda)$.

The objective function is the weighted least-squares method

$$Y=\sum_{k=1}^{N}\left(\frac{l-l_k^\circ}{\sigma_k^\circ}\right)^2. \tag{3.4}$$

N : number of data

l_k°, σ_k° : data

Then it leads to

$$\frac{\partial Y}{\partial l}=\sum_k\frac{2(l-l_k^\circ)}{\sigma_k^{\circ 2}},\quad \frac{\partial^2 Y}{\partial l^2}=\sum_k\frac{2}{\sigma_k^{\circ 2}}, \tag{3.5}$$

$$\frac{\partial^2 Y}{\partial\theta_i\partial\theta_j}=\frac{\partial^2 Y}{\partial l^2}\cdot\frac{\partial l}{\partial\theta_i}\cdot\frac{\partial l}{\partial\theta_j}+\frac{\partial Y}{\partial l}\cdot\frac{\partial^2 l}{\partial\theta_i\partial\theta_j}. \tag{3.6}$$

Because the second term of (3.6) contributes a little, it is omitted generally. Then we get

$$\frac{\partial^2 Y}{\partial\theta_i\partial\theta_j}\doteqdot\frac{\partial^2 Y}{\partial l^2}\cdot\frac{\partial l}{\partial\theta_i}\cdot\frac{\partial l}{\partial\theta_j}=\sum_{k=1}^{N}\frac{2}{\sigma_k^{\circ 2}}\cdot\frac{\partial l}{\partial\theta_i}\cdot\frac{\partial l}{\partial\theta_j}. \tag{3.7}$$

Therefore, the diagonal elements of $\boldsymbol{H}$ are always positive.

2. Partial differentiation by parameters

Parameters of RICHARDS model (2.8) and (2.22) are l_∞, K, t_0, t_1, a and n. Concrete expressions of $\partial l/\partial\theta$ are as follows :

$$\frac{\partial l}{\partial l_\infty}=\frac{l}{l_\infty}. \tag{3.8}$$

$$\frac{\partial l}{\partial\theta}=-l\,\frac{e^h}{1+ne^h}\cdot\frac{\partial h}{\partial\theta},\quad \theta=K,\ t_0,\ t_1,\ a. \tag{3.9}$$

Where for (2.8)

$$\frac{\partial h}{\partial K}=-(t-t_0)$$

$$\frac{\partial h}{\partial t_0}=K.$$

Where for (2.22)

$$\frac{\partial h}{\partial K}=-\{F(t)-F(t_0)\}$$

$$\frac{\partial h}{\partial t_0}=Kf(t_0)$$

$$\frac{\partial h}{\partial t_1}=-K\left\{\frac{\partial F(t)}{\partial t_1}-\frac{\partial F(t_0)}{\partial t_1}\right\}$$

$$\frac{\partial F(t)}{\partial t_1}=-\frac{1-a}{2}\cos 2\pi(t-t_1)$$

$$\frac{\partial h}{\partial a}=-K\left\{\frac{\partial F(t)}{\partial a}-\frac{\partial F(t_0)}{\partial a}\right\}$$

$$\frac{\partial F(t)}{\partial a}=\frac{1}{2}t-\frac{1}{4\pi}\sin 2\pi(t-t_1).$$

For n

$$\frac{\partial l}{\partial n}=l\frac{1}{n}\left\{\frac{1}{n}\ln(1+nx)-\frac{x}{1+nx}\right\}>0,\quad x=e^{h}. \tag{3.10}$$

$$\lim_{n\to 0}\frac{\partial l}{\partial n}=l\frac{x^2}{2}>0.$$

The sign of (3.10) is apparent in Fig. 2.

3. Correction of the "LOG" function

When $n\to 0$ it is difficult to calculate (2.8), (2.22) and (3.10) precisely. When we use a high precision computer, it is sufficient to be careful only for $n=0$. The probability of n being 0 is so low for normal data that we can ignore this case. But, when we use a low precision computer, this problem is important because the precision of its "LOG" function is too low.

TAKAHASHI (1974) and HITOTSUMATSU (1981) suggested that computers treat the calculation of power as follows :

$$x^y\to \mathrm{EXP}(y*\mathrm{LOG}(x)).$$

Although the "EXP" function is high precision, the "LOG" function is low precision. The values of $\ln(1+n)$ for a personal computer PC9801F (NEC, N_{88}-BASIC) and a hand-held calculator FX502P (CASIO) are shown in Table 1. The FX502P result is correct and the PC9801F result is not correct because of TAYLOR series as follows :

$$\ln(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}-\cdots\cdots \tag{3.11}$$

Table 1. Values of LOG$(1+n)$ for PC9801F (NEC) and FX502P (CASIO).

n	PC9801F	FX502P
100000	11.5129	11.51293546
10000	9.21044	9.210440367
1000	6.90875	6.908754779
100	4.61512	4.615120516
10	2.3979	2.397895272
1	0.693147	0.69314718
0.1	9.53102×10^{-2}	9.5310179×10^{-2}
0.01	9.95025×10^{-3}	9.950330853×10^{-3}
0.001	9.99446×10^{-4}	9.995003330×10^{-4}
10^{-4}	9.99405×10^{-5}	9.999500033×10^{-5}
10^{-5}	9.91555×10^{-6}	9.999950000×10^{-6}
10^{-6}	9.08925×10^{-7}	9.999995000×10^{-7}
10^{-7}	8.26296×10^{-8}	9.999999500×10^{-8}
10^{-8}	0	9.999999950×10^{-9}
10^{-9}	0	$9.999999995\times10^{-10}$

And (3.10) has the problem of cancellation. Let

$$a=\frac{1}{n}\ln(1+nx),\quad b=\frac{x}{1+nx}.$$

When $n=0.1$, $x=0.005$ it is as follows :

	PC9801F	FX502P
a	4.99826	4.998750416
b	4.9975	4.997501249
$a-b$	0.00076	0.001249167

Thus, the PC9801F gives the wrong values. When n is small, the number of significant figures drop even in the FX502P values.

In this paper, for (2.8) and (2.22) the following expression is used by (3.11).

$$(1+nx)^{\frac{1}{n}}=\exp\left[x\left\{1-\frac{nx}{2}+\frac{(nx)^2}{3}-\frac{(nx)^3}{4}+\cdots\cdots\right\}\right] \tag{3.12}$$

And for (3.10) the following expression is used by (3.11) and

$$\frac{1}{1+z}=1-z+z^2-z^3+z^4-\cdots\cdots. \tag{3.13}$$

$$\frac{\partial l}{\partial n}=l\frac{x^2}{2}\left\{1-\frac{2\cdot 2nx}{3}+\frac{2\cdot 3(nx)^2}{4}-\frac{2\cdot 4(nx)^3}{5}+\cdots\cdots\right\},$$

$$x=e^h. \tag{3.14}$$

(3.12) and (3.14) are used when $|nx|\leq 0.1$.

IV. AIC

When we treat n as a parameter the number of parameters increases by 1. Therefore, the likelihood increases. But, the confidence area will be enlarged because the correlations of parameters increase. An adequate number of parameters will be presented by AIC (AKAIKE information criterion) :

$$\mathrm{AIC}=-2\ln L_{\max}+2r \tag{4.1}$$

$L_{\max}$: maximum likelihood
r : number of parameters

For the weighted least-squares method (3.4), $(l_k-l_k^\circ)/\sigma_k^\circ$ distributes according to $N(0,1)$. Then it leads to

$$L=\prod_{k=1}^{N}P_k=\left(\frac{1}{\sqrt{2\pi}}\right)^N\exp\left(-\frac{1}{2}Y\right), \tag{4.2}$$

$$\mathrm{AIC}=Y_{\min}+2r+\mathrm{const.} \tag{4.3}$$

On the other hand, for the least-squares method :

$$Y^* = \sum_{k=1}^{N} (l - l_k^\circ)^2 \qquad (4.4)$$

$(l_k - l_k^\circ)$ distributes according to N $(0, \sigma)$. Then it ieads to

$$L = \left(\frac{1}{\sqrt{2\pi}\ \sigma}\right)^N \exp\left(-\frac{1}{2}\ \frac{Y^*}{\sigma^2}\right). \qquad (4.5)$$

The following is used for the σ^2 estimate.

$$\sigma^2 = \frac{Y^*_{\min}}{N - r} \qquad (4.6)$$

Then it leads to

$$\text{AIC} = N \ln Y^*_{\min} + 2r + \text{const.} \qquad (4.7)$$

Where the following approximation is used.

$$\ln\left(1 - \frac{r}{N}\right) \doteqdot -\frac{r}{N} \qquad (r \ll N)$$

The model which minimizes AIC is regarded as the best. Namely, the model which explains the data efficiently with fewer parameters is regarded as best. Generally, AIC may be useful in the condition $r \leqq 2\sqrt{N}$.

V. Computer program

The BASIC programs for PC9801F (NEC) are listed in Appendices B and C. These are the only changing parts from AKAMINE (1986)'s program 1. When $n<0$, the data satisfying $1 + ne^h \leqq 0$ are cut off. This is not so important in practice. After line 20000 there is a correction to the "LOG" function. This is not necessary for high precision computers or languages. It is better to check the "LOG" function before use.

VI. Experiments

AKAMINE (1986)'s data (Table 2) is used for the test of these programs and results are shown in Table 3. Adding n to the parameters makes $Y_{\min}$ smaller but AIC larger. Namely, it is not efficient for these data. It is natural because they are made for $n = -1$, 0 and 1. On the other hand, adding t_1 and a to the parameters makes AIC smaller. They relate only to oscillation.

Acknowledgements

The author is indebted to Mr. K. ISHIOKA of Tokai Regional Fisheries Research Laboratory and Mr. F. KATO of Seikai Regional Fisheries Research Laboratory for their kind advice and help. The author is also grateful to Mr. K. NOGAMI and Dr.

K. TANAKA of Japan Sea Regional Fisheries Research Laboratory for their critical reading of the manuscript.

Table 2-a. The data for the experiment (data-1).

k	t_k	l_k°	σ_k°	k	t_k	l_k°	σ_k°	k	t_k	l_k°	σ_k°
1	0.5	5	3	8	2.0	47	2	15	3.5	80	3
2	0.8	12	3	9	2.2	54	3	16	4.0	82	2
3	1.0	18	2	10	2.4	63	3	17	4.5	87	3
4	1.2	30	4	11	2.5	66	3	18	5.0	88	3
5	1.3	36	3	12	2.8	69	3	19	7.5	99	5
6	1.5	42	3	13	3.0	68	6	20	10.0	99	2
7	1.7	45	3	14	3.2	74	3				

Table 2-b. The data for the experiment (data-2).

k	t_k	l_k°	σ_k°	k	t_k	l_k°	σ_k°	k	t_k	l_k°	σ_k°
1	−4.0	1	2	10	−0.7	38	4	19	2.0	87	3
2	−3.5	3	2	11	−0.3	39	5	20	2.3	92	4
3	−3.0	4	3	12	0.0	48	4	21	2.8	93	3
4	−2.8	7	3	13	0.3	61	4	22	3.0	96	3
5	−2.4	7	3	14	0.6	62	4	23	3.5	97	2
6	−2.0	13	4	15	0.8	66	5	24	4.0	98	3
7	−1.7	17	3	16	1.0	75	4	25	5.0	99	3
8	−1.4	18	3	17	1.3	81	3				
9	−1.0	25	4	18	1.7	82	3				

Table 2-c. The data for the experiment (data-3).

k	t_k	l_k°	σ_k°	k	t_k	l_k°	σ_k°	k	t_k	l_k°	σ_k°
1	−1.3	1	3	8	0.2	50	2	15	2.0	88	2
2	−1.0	8	3	9	0.4	55	3	16	2.3	92	3
3	−0.8	16	2	10	0.7	56	5	17	2.7	92	3
4	−0.5	19	3	11	1.0	67	3	18	3.0	94	3
5	−0.3	24	3	12	1.2	77	3	19	4.0	98	2
6	−0.1	29	4	13	1.5	81	2	20	5.0	98	3
7	0.0	40	4	14	1.8	82	3				

Table 3. Results of experiments. Comparison with AKAMINE (1986).

	Number of iterations	l_∞	K	t_0	t_1	a	n	Y_{min}	r[a]	AIC[b]
data-1	0	100	.5	.5	.25	0	−1			
	3	99.9635	.507699	.558275	——	(1)	(−1)	15.7711[c]	3	21.7711
	6	100.139	.496821	.504319	——	(1)	−1.03834	15.7461[c]	4	23.7461
	5	100.623	.870266	.388283	.229144	.119223	(−1)	3.93503	5	13.93503
	10	100.225	.906215	.475398	.223683	.124174	−.929432	3.81993	6	15.81993
data-2	0	100	1.0	0	0	0	0			
	4	99.1726	.993326	−.080152	——	(1)	(0)	17.8405	3	23.8405
	7	99.1302	.997562	−.075764	——	(1)	−.008186	17.8391	4	25.8391
	9	99.7023	1.94285	−.014149	.006776	.015631	(0)	2.65882	5	12.65882
	9	99.4431	2.0074	−.002191	.007257	.011215	.058150	2.60802	6	14.60802
data-3	0	100	1.0	0	0	0	1			
	5	99.7154	1.00196	−.003079	——	(1)	(1)	7.28418	3	13.27418
	5	99.949	.975934	−.028745	——	(1)	.938929	7.25647	4	15.25647
	7	99.7416	2.13245	.010413	.014517	.050024	(1)	1.27517	5	11.27517
	7	99.868	2.10076	.003795	.014613	.049559	.966074	1.26659	6	13.26659

a) number of parameters
b) $AIC = Y_{min} + 2r$
c) $k=1$ data is cut off.

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RICHARDS の式のパラメータ推定

赤 嶺 達 郎

AKAMINE (1986) の MARQUARDT 法の BASIC プログラムを RICHARDS の式およびその周期関数による拡張式用に書き換えた．0.9～1.1において TAYLOR 級数で "LOG" 関数の修正を行った．推定値が負となるデータは除外した．n をパラメータに加える事の妥当性を AIC で判定した．RICHARDS の式は応用上はあまり重要ではないが，理論上重要である．

Appendix A. Partial defferentiation by n for (2.10).

This is as follows :

$$\frac{\partial l}{\partial n}=l\frac{1}{n}\left[\frac{1}{n}\ln\{1+(p^n-1)x\}-\frac{xp^n\ln p}{1+(p^n-1)x}\right]<0,\ x=e^h.$$

$$\lim_{n\to 0}\frac{\partial l}{\partial n}=-(\ln p)^2\frac{x-x^2}{2}<0.$$

The sign is apparent in Fig. 1. These expressions seem to be more difficult to treat than (3.10). The expression of $n\to 0$ is led by (3.11), (3.13) and

$$\lim_{n\to 0}\frac{\dfrac{y^n-1}{n}-\ln y}{n}=\frac{(\ln y)^2}{2}.$$

In addition to

$$\lim_{n\to 0}\frac{e^x-(1+nx)^{\frac{1}{n}}}{n}=\frac{e^x x^2}{2}.$$

These are led by the following theory :

$$\text{When }\lim_{n\to a}f=0\text{ and }\lim_{n\to a}g=0,\ \lim_{n\to a}\frac{f}{g}=\lim_{n\to a}\frac{f'}{g'}.$$

Appendix B. The BASIC program to estimate parameters for RICHARDS model. Changing parts from program-1 of AKAMINE (1986).

```
10      '--------------------------------------------
20      '    Richards model by Marquardt's method
30      '
40      '                  by Tatsuro Akamine
50      '                               1987-06-24
60      '--------------------------------------------
1085    '-------------------------------
1090    '    Definition of functions
1095    '-------------------------------
1100     NP=4
1110     DEF FNEP1=EXP(-P2(2)*(TIME(K)-P2(3)))
1115     DEF FNEP2=1+P2(4)*FNEP1
1120     DEF FNDP1=1/POWER9
1130     DEF FNBL =P2(1)*FNDP1
1140     DEF FNDP2=FNBL*FNEP1/FNEP2*(TIME(K)-P2(3))
1150     DEF FNDP3=-FNBL*FNEP1/FNEP2*P2(2)
1190     DEF FNDP4=FNBL*DLDN9
1200     DEF FNPC9801=FNEP2^(1/P2(4))
1210     DEF FNPC9802=(LOG(FNEP2)/P2(4)-FNEP1/FNEP2)/P2(4)
4072    '--------------------------------------
4073    '    Check for 1+nx>0 and correction
4074    '--------------------------------------
4075       IF FNEP2<=0 THEN PRINT "CANCEL 2 I=";K : GOTO *CSKIP2
4078       GOSUB *CHECK1
4105       GOSUB *CHECK2
4106       DIFFER(4)=FNDP4
4160    NEXT J : NEXT I
4163         *CSKIP2
4165    NEXT K
6023       IF FNEP2<=0 THEN LPRINT "CANCEL 1 I=";K : GOTO *CSKIP1
6025       GOSUB *CHECK1
6035      *CSKIP1
8065    PRINT "     Richards  n     =";P(4)
19985 '--------------------------------------------
19990 '    Correction of (1+nx)^(1/n) and dl/dn
19995 '--------------------------------------------
20000 *CHECK1
20010   BRANCH=P2(4)*FNEP1
20020   IF ABS(BRANCH)>.1 THEN POWER9=FNPC9801 ELSE GOSUB *CORRECT1
20030  RETURN
20100 *CHECK2
20110   BRANCH=P2(4)*FNEP1
20120   IF ABS(BRANCH)>.1 THEN DLDN9=FNPC9802 ELSE GOSUB *CORRECT2
20130  RETURN
21000 *CORRECT1
21010   CORY1=-BRANCH
21020   CORI1=2 : CORD1=1
21030  *CORSTART1
21040    CORC1=CORY1/CORI1
21050    IF ABS(CORC1)<.0000001 THEN *COREND1
21060    CORD1=CORD1+CORC1
21070    CORY1=CORY1*(-BRANCH) : CORI1=CORI1+1
21080    GOTO *CORSTART1
21090  *COREND1
21100    POWER9=EXP(FNEP1*CORD1)
21110  RETURN
22000 *CORRECT2
22010   CORY2=-BRANCH
22020   CORI2=3 : CORD2=1
22030  *CORSTART2
22040    CORC2=2*CORY2*(CORI2-1)/CORI2
22050    IF ABS(CORC2)<.0000001 THEN *COREND2
22060    CORD2=CORD2+CORC2
22070    CORY2=CORY2*(-BRANCH) : CORI2=CORI2+1
22080    GOTO *CORSTART2
22090  *COREND2
22100    DLDN9=FNEP1*FNEP1*CORD2/2
22110  RETURN
```

Appendix C. The BASIC program to estimate parameters for Richards model expanded by a periodic function. Changing parts from appendix B.

```
10      '-----------------------------------------------------
20      '    Richards model expanded by periodic function
30      '    by Marquardt's method
40      '                              by Tatsuro Akamine
50      '                                          1987-06-25
60      '-----------------------------------------------------
1020     PAI=3.14159265#
1100     NP=6
1110     DEF FNEP1=EXP(-P2(2)*(FNFT1(TIME(K))-FNFT1(P2(3))))
1115     DEF FNEP2=1+P2(4)*FNEP1
1120     DEF FNDP1=1/POWER9
1130     DEF FNBL =P2(1)*FNDP1
1140     DEF FNDP2=FNEP3*(FNFT1(TIME(K))-FNFT1(P2(3)))
1150     DEF FNDP3=-FNEP3*P2(2)*FNFT3(P2(3))
1190     DEF FNDP4=FNBL*DLDN9
1200     DEF FNDP5=FNEP3*P2(2)*(FNFT4(TIME(K))-FNFT4(P2(3)))
1210     DEF FNDP6=FNEP3*P2(2)*(FNFT5(TIME(K))-FNFT5(P2(3)))
1300     DEF FNEP3=FNBL*FNEP1/FNEP2
1400     DEF   FNFT1(TM)=FNP51*TM+FNP52/2/PAI*SIN(FNTM1(TM))
1410     DEF   FNFT3(TM)=FNP51+FNP52*COS(FNTM1(TM))
1420     DEF   FNFT4(TM)=-FNP52*COS(FNTM1(TM))
1430     DEF   FNFT5(TM)=TM/2-1/4/PAI*SIN(FNTM1(TM))
1450     DEF     FNP51=(1+P2(6))/2
1460     DEF     FNP52=(1-P2(6))/2
1470     DEF     FNTM1(TM)=2*PAI*(TM-P2(5))
4102       DIFFER(5)=FNDP5
4103       DIFFER(6)=FNDP6
8066    PRINT "            T1      =";P(5)
8067    PRINT "            A       =";P(6)
```